Sample Solution

DEPARTMENT OF MATHEMATICAL SCIENCES

MATH 301 Methods of Applied Mathematics Term 061

QUIZ # 1(a)

Name ______________________ ID # __________ Section # ____

Q1) (a) Find the parametric form of curve of intersection given by $z = x^2 + y^2, z = 4, x = 2\sin t$.

$x = 2\sin t$ and $z = 4$ gives $4 = y^2 + 4\sin^2 t \Rightarrow y = 2\cos t$

Thus $x = 2\sin t$; $y = 2\cos t$; $z = 4$ is parametric form

This can be written as

$\underline{r}(t) = x\underline{i} + y\underline{j} + z\underline{k} = 2\sin t\,\underline{i} + 2\cos t\,\underline{j} + 4\underline{k}$

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(b) Find the tangent vector to above curve at point P given by $t = \dfrac{\pi}{4}$ and put it in unit vector form.

The tangent is along $\underline{r}'(t)$ at any point Hence

$\underline{r}'(t) = 2\cos t\,\underline{i} - 2\sin t\,\underline{j} + 0\,\underline{k}$ is direction of tangent

$\underline{a} = \underline{r}'(t)\big|_{t=\frac{\pi}{4}} = 2\cos\frac{\pi}{4}\,\underline{i} - 2\sin\frac{\pi}{4}\,\underline{j} + 0\,\underline{k} = 2\cdot\frac{1}{\sqrt{2}}\,\underline{i} - 2\frac{1}{\sqrt{2}}\,\underline{j} + 0\,\underline{k}$

$= \sqrt{2}\,\underline{i} - \sqrt{2}\,\underline{j} + 0\,\underline{k}$

$\|\underline{a}\| = \sqrt{2 + 2 + 0} = 2$

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Thus unit tangent $\underline{u} = \dfrac{\underline{a}}{\|\underline{a}\|} = \dfrac{1}{\sqrt{2}}\,\underline{i} - \dfrac{1}{\sqrt{2}}\,\underline{j} + 0\,\underline{k}$

(c) Find directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in the direction of above tangent vector at the given point P.

$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$

$\underline{\nabla} f = \langle 2x, 2y, 2z \rangle$. At $t = \frac{\pi}{4}$; $x = 2\sin\frac{\pi}{4} = \sqrt{2}$, $y = 2\cos\frac{\pi}{4} = \sqrt{2}$, $z = 4$

$\underline{\nabla} f\Big|_{(\sqrt{2}, \sqrt{2}, 4)} = \langle 2\sqrt{2}, 2\sqrt{2}, 8 \rangle$

(4)

$D_{\underline{u}} f = \underline{\nabla} f \cdot \underline{u} = \langle 2\sqrt{2}, 2\sqrt{2}, 8 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \rangle$

$= 2 - 2 = 0$

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QUIZ # 1(b)

Name ______________________ ID # __________ Section # ____

Q1) (a) Find the parametric form of curve $x^2 + 4y^2 = 1$.

An appropriate parametric form is

$$\left.\begin{aligned} x &= \cos t \\ y &= \tfrac{1}{2}\sin t \end{aligned}\right\} \quad x^2 + 4y^2 = \cos^2 t + 4\left(\tfrac{1}{4}\right)\sin^2 t = 1$$

Thus $\quad \underline{r}(t) = x\underline{i} + y\underline{j} = \cos t\,\underline{i} + \frac{1}{2}\sin t\,\underline{j}$

(b) Find the tangent vector to above curve at $P(\frac{\sqrt{3}}{2}, \frac{1}{4})$ and put it in unit vector form.

$$\text{Given}\quad \left.\begin{aligned} x &= \frac{\sqrt{3}}{2} = \cos t \\ y &= \frac{1}{4} = \frac{1}{2}\sin t \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos t &= \frac{\sqrt{3}}{2} \\ \sin t &= \frac{1}{2} \end{aligned}\right\} \Rightarrow t = \frac{\pi}{6}$$

Tangent vector : $\underline{r}'(t) = -\sin t\,\underline{i} + \frac{1}{2}\cos t\,\underline{j}$

$$\underline{r}'(t)\Big|_{t=\frac{\pi}{6}} = -\sin\frac{\pi}{6}\,\underline{i} + \frac{1}{2}\cos\frac{\pi}{6}\,\underline{j} = -\frac{1}{2}\,\underline{i} + \frac{\sqrt{3}}{4}$$

$\|r'(t)\| = \sqrt{\frac{1}{4} + \frac{3}{16}} = \frac{\sqrt{7}}{4}$. Thus unit vector $= \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{4} \right\rangle \frac{4}{\sqrt{7}}$

$$= \left\langle \frac{-2}{\sqrt{7}}, \sqrt{\frac{3}{7}} \right\rangle$$

(c) Find directional derivative of $f(x, y) = x^2 + 2y^2$ in the direction of above tangent vector.

$$\underline{\nabla} f = \langle f_x, f_y \rangle = \langle 2x, 4y \rangle$$

$$x = \frac{\sqrt{3}}{2}, \quad y = \frac{1}{4}$$

$$\underline{\nabla} f\Big|_P = \left\langle 2\frac{\sqrt{3}}{2}, 4\left(\frac{1}{4}\right) \right\rangle = \langle \sqrt{3}, 1 \rangle$$

$$D_u f = \nabla f \cdot \underline{u} = \langle \sqrt{3}, 1 \rangle \cdot \left\langle \frac{-2}{\sqrt{7}}, \sqrt{\frac{3}{7}} \right\rangle$$

$$= -2\sqrt{\frac{3}{7}} + \sqrt{\frac{3}{7}} = -\sqrt{\frac{3}{7}}.$$

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QUIZ # 1(c)

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Q1) Find the arc length of the curve $x = e^t \cos 2t, y = e^t \sin 2t, z = e^t, 0 \le t \le 2\pi$.

$$\frac{dx}{dt} = e^t \cos 2t - 2e^t \sin 2t \; ; \left(\frac{dx}{dt}\right)^2 = e^{2t}\cos^2 2t + 4e^{2t}\sin^2 2t - 4e^{2t}\cos 2t \sin 2t$$

$$\frac{dy}{dt} = e^t \sin 2t + 2e^t \cos 2t \; ; \left(\frac{dy}{dt}\right)^2 = e^{2t}\sin^2 2t + 4e^{2t}\cos^2 2t + 4e^{2t}\sin 2t \cos 2t$$

$$\frac{dz}{dt} = e^t \; ; \left(\frac{dz}{dt}\right)^2 = e^{2t}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = e^{2t}(\cos^2 2t + \sin^2 2t) + 4e^{2t}(\sin^2 2t + \cos^2 2t) + e^{2t} = 6e^{2t}$$

$$s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{6}\int_0^{2\pi} e^t\, dt = \sqrt{6}\,(e^{2\pi} - 1). \quad (4)$$

Q2)(a) Find a vector giving the direction of most rapid decrease of the function $f(x, y, z) = \ln \frac{yz}{x}$, at $P(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})$. What is the rate of most rapid decrease?

$-\nabla f$ is direction of most rapid decrease where $\nabla f = \langle f_x, f_y, f_z \rangle$

$$f_x = \frac{x}{yz}\left(\frac{-yz}{x^2}\right) = -\frac{1}{x}, \quad f_y = \frac{x}{yz}\left(\frac{z}{x}\right) = \frac{1}{y} \text{ and } f_z = \frac{x}{yz}\frac{y}{x} = \frac{1}{z}$$

so that $-\nabla f = -\langle -\frac{1}{x}, \frac{1}{y}, \frac{1}{z} \rangle = \langle \frac{1}{x}, -\frac{1}{y}, -\frac{1}{z} \rangle$

$$-\nabla f \Big|_{P(\frac{1}{3}, \frac{1}{6}, \frac{1}{2})} = \langle 3, -6, -2 \rangle$$

$$\text{Rate} = \|\nabla f\| = \sqrt{9 + 36 + 4} = 7. \quad (3)$$

(b) Find directional derivative of the above *f(x,y,z)* in the direction of vector from *(1,4,5)* to *(2,5,4)*

Required direction $= \langle 2-1, 5-4, 4-5 \rangle = \langle 1, 1, -1 \rangle$

$\underline{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$ (we need unit vector)

$$D_u f = \nabla f \cdot \underline{u} = \langle -3, 6, 2 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$$

$$= -\sqrt{3} + \frac{6}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\sqrt{3} + 2\sqrt{3} - \frac{2}{\sqrt{3}}$$

$$= \sqrt{3} - \frac{2}{\sqrt{3}} = \frac{+1}{\sqrt{3}} \quad (3)$$