On Perturbation and Beyond

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Abstract

Perturbation method has been a powerful tool in solving problems arising in mathematical physics, direct and inverse scattering, solid and fluid dynamics and various engineering disciplines. After a brief introduction to the method and some interesting applications, some new alternates will be discussed. In particular a brief account of the decomposition method introduced by Adomian and homotopy analysis method recently proposed by Shijun Liao will be provided.

Plan of the talk

- Basic Idea
- Boundary value Problems
- Eigenvalue Problems
- Adomian Method
- Homotopy Analysis Method
- Concluding Remarks

- Consider an initial or boundary value problem

  $Au = f$, where A is some differential operator, acting on a domain that lies in an appropriate Hilbert space and f is specified.

  For even simple differential operators, closed form solution may not be available except for some nice values of f.
- The idea is to closely approximate the given problem to that which can be solved exactly. This approximation is achieved by introducing a small parameter ε
- Secondly it is assumed that the solution to the given problem can be written as a power series in ε(called perturbation expansion)
- Next, the expansion is put in the given problem and coefficients of powers of ε are compared.
- Zeroth order problem corresponds to un-perturbed case and provides with the leading term in the solution

Perturbed BVP

Consider the equation

$$u'' + (1+\varepsilon x^2)u = f(x),\ 0 < x < 1,$$

$$u(0) = u(1) = 1.$$

The unperturbed problem corresponds to $\varepsilon = 0$

$$u'' + u = 0, \qquad 0 < x < 1$$

$$u(0) = u(1) = 1$$

If g(x,ξ) is Green's function of the problem, the solution to the unperturbed problem can be written as

where $w_0(x)$ is solution of the homogeneous problem.
We can easily find the relevant Green's function.
The coefficient of ε gives rise to the inhomogeneous problem

$$u_1^{\prime\prime} + u_1 = x^2 u_0,$$

$$u_1(0) = u_1(1) = 0$$

$$Put \quad F(x) = \int_0^1 g(x,\xi) f(\xi) d\xi$$

We can then write the solution as

$$u(x) = w_0(x) + F(x) + \int_0^1 g(x,\xi)\xi^2\left[w_0(\xi) + F(\xi)\right]d\xi$$

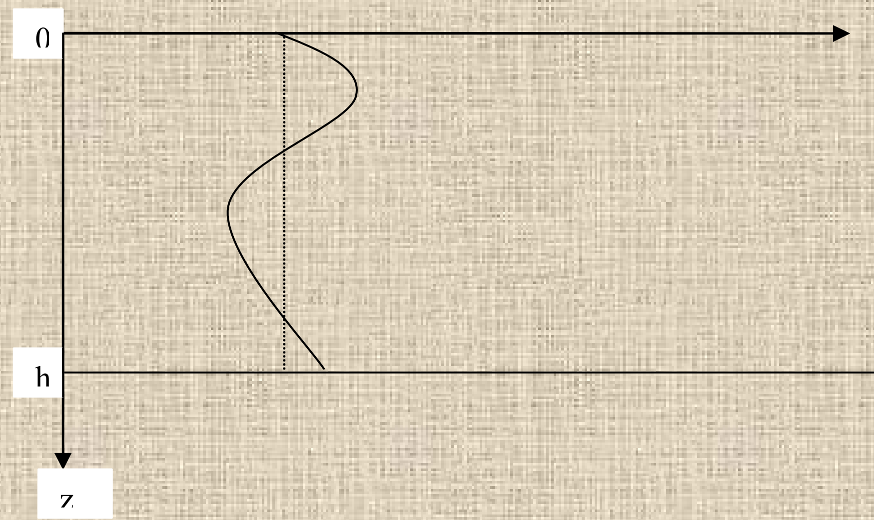
This simple idea can be used in direct and inverse scattering problems in which the physical parameters (properties) of a medium show small determinate or random variations from a background comparison medium. Some examples are

- Zaman, Asghar and Ahmad: Dispersion of Love type waves in an inhomogeneous layer, J. Phys Earth, 1990.
- Zaman and Masood: Recovery of propagation speed and damping of the medium, Il Nuovo Cimento (C) , 2002.
- Zaman and Al –Zayer : Dispersion of Love waves in a stochastic layer, Il Nuovo Cimento (To Appear).

Eigenvalue Problem

- Find eigenvalues and eigenfunctions of the unperturbed problem
- If the unperturbed problem is symmetric. The eigenvalues are real and eigenfunctions form a complete orthonormal set
- This provides a powerful tool to recover the correction terms in the perturbed eigenvalues
- The perturbed eigenfunctions can only be obtained as a Fourier series

- Problem in Ocean Acoustics



$$p_{rr}(r,z)+\frac{1}{r}p_r(r,z)+p_{zz}(r,z)+k^2p(r,z)=0$$

$p(r,0) = 0$ and $p_z(r,h) = 0$.

We solve using separation of variables

$$p(r,z)=\Phi(r)\theta(z)$$

- The Idealized Depth Equation is the Sturm Liouville Problem

$$\phi^{//}(z)+k^2\phi(z)=\lambda\phi(z)$$

$$\phi(0)=0 \quad and \quad \phi^{/}(h)=0$$

The normalized eigenfunctions and eigenvalues are

$$\phi_m(z)=\sqrt{\frac{h}{2}}\sin\frac{(2m-1)\pi z}{2h}$$

$$\lambda_m=k^2-\left[\frac{(2m-1)}{2h}\right]^2$$

$$m=1,2,3\;..........\qquad .....$$

An ocean with depth dependent properties leads to

$$\psi''(z) + k^2 n^2(z)\psi(z) = \lambda \psi(z)$$

with boundary conditions

$$\psi(0) = 0 \quad and \quad \psi'(h) = 0$$

The index of refraction

$$n^2(z) = 1 + \varepsilon s(z)$$

Contains a perturbation term $\varepsilon s(z)$

The case $\varepsilon = 0$ represents an idealized ocean

Perturbation Results

The first correction term in the perturbation series can be found as

$$\lambda_m^{(1)} = \frac{2k^2}{h}\int_0^h s(z)\left[\sin\frac{(2m-1)\pi z}{2h}\right]^2 dz$$

The Fourier coefficients are

$$\alpha^{(1)}{}_{mn} = \frac{2k^2}{h}\int s(z)\sin\frac{(2m-1)\pi z}{2h}\sin\frac{(2n-1)\pi z}{2h}dz$$

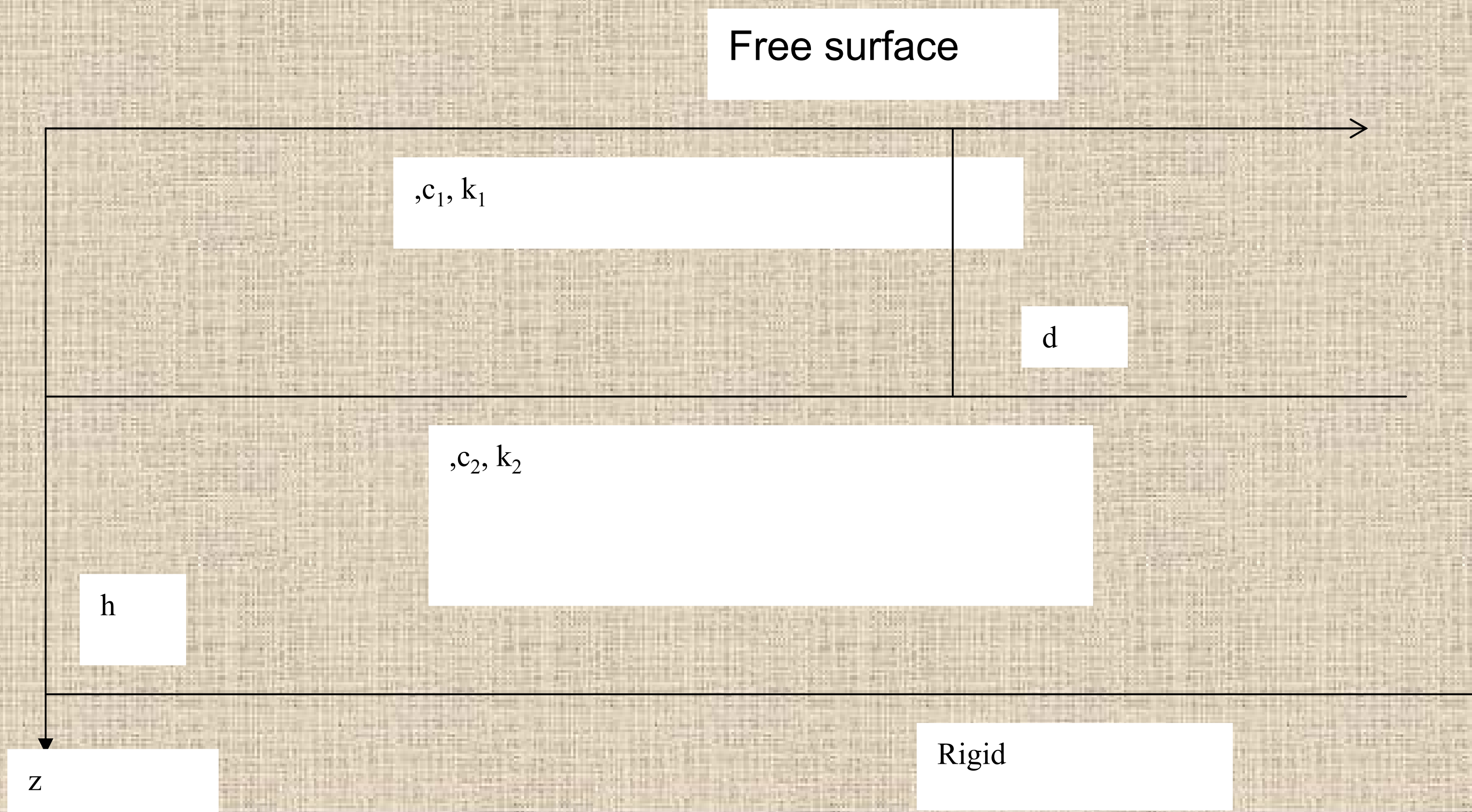
$$\alpha^{(1)}{}_{mm} = 0$$

$$\psi_m(z)^{(1)} = \sum_1^{\infty}\alpha^{(1)}{}_{mk}\,\phi_k(z)$$

Layered Model (Zaman &Al-Muhiameed Applied Acoustics:2000)

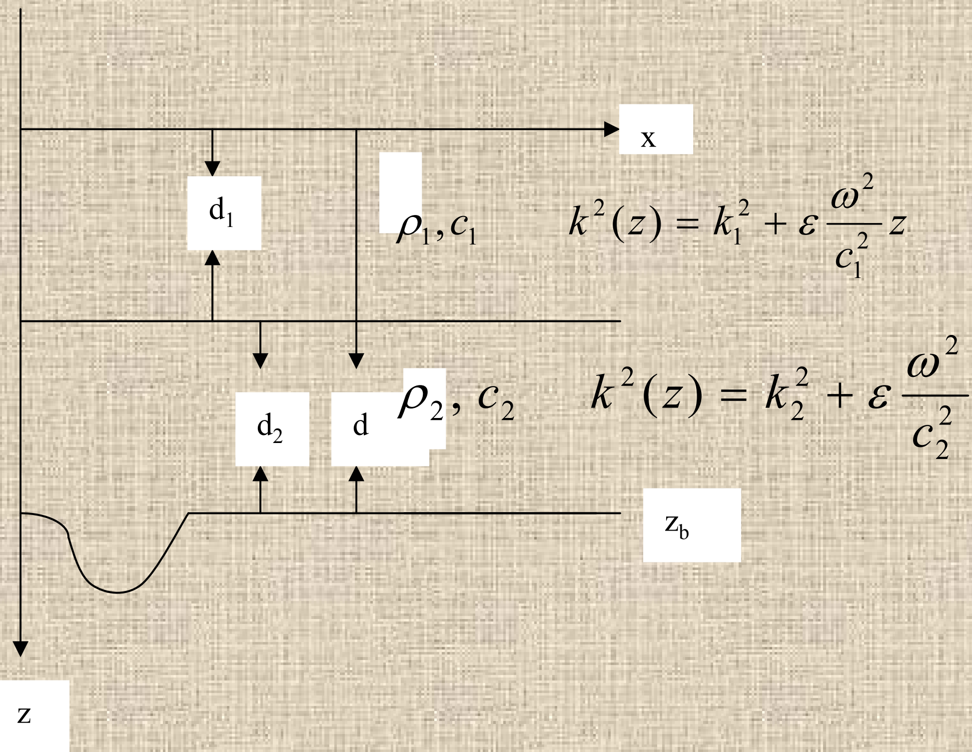
- Ocean properties are depth dependent
- The variations is piecewise constant
- Depth equation has piecewise constant coefficients

- Rigid seabed assumption gives nice normal mode theory



Undulated Seabed (Zaman & Marzoug ICIAM 2003)

Inhomogeneous, layered model with non-smooth seabed



Beyond Perturbation Methods

- Decomposition Method

Adomian, G., Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, London &. Boston, 1994.

- Homotopy Analysis Method

Liao, Shijun, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman & Hall/ CRC, 2004.

Adomian Method

- Powerful tools for ordinary and partial differential equations
- Can be applied to nonlinear problems
- The approximation series converges quickly
- Approximate solutions by this method often are in the terms of polynomials
- The method is based upon expressing the solution in terms of inverse operator of a linear operator and then approximating it by an infinite series

Method Description

Consider the nonlinear problem

$$M\left[u\left(\underline{r}, t\right)\right] = f\left(\underline{r}, t\right)$$

where M is a nonlinear operator, u is a dependent variable, $f(\underline{r}, t)$ is a known function, and $\underline{r}$ and t are spatial and temporal variables. Assume that the nonlinear operator can be expressed as

$$M = L_0 + N_0$$

$$\textit{where } L_0 \textit{ is a linear operator and } N_0 \textit{ is some nonlinear operator}$$

Under these assumptions the original problem can be written as

$$L_0\left[u\left(\underline{r}, t\right)\right] + N_0\left[u\left(\underline{r}, t\right)\right] = f\left(\underline{r}, t\right)$$

The solution is then expressed as

$$u(\underline{r},t) = u_0(\underline{r},t) + \sum_1^{\infty} u_n(\underline{r},t)$$

where

$$u_0(\underline{r},t) = L_0^{-1}\left[f(\underline{r},t)\right]$$

$$u_n(\underline{r},t) = -L_0^{-1}\left[A_{n-1}(\underline{r},t)\right]$$

$$A_n(\underline{r},t) = \frac{1}{n!}\left[\frac{d^n}{dq^n} N_0\left(u_0(\underline{r},t) + \sum_1^{\infty} u_n(\underline{r},t)q^n\right)\right]_{q=0}$$

$$u_1(\underline{r},t) = -L_0^{-1}\left[A_0(\underline{r},t)\right]$$

$$A_0(r,t) = N_0\left[u_0(r,t)\right]$$

Illustration

Consider the problem

$$V'(t) + V^2(t) = 1, \quad t \geq 0$$

$$V(0) = 0$$

We can replace this equation by the following

$$V(t) = t - \int_0^t V^2(t)\,dt$$

The solution is given by

$$V(t) = V_0(t) + \sum_1^\infty V_k(t)$$

where

$$V_0(t) = t$$

$$V_k(t) = -\int_0^t A_{k-1}(t)dt, \quad k \geq 1$$

$$A_k(t) = \sum_0^k V_n(t)V_{k-n}(t)$$

$$where$$

$$V_1(t) = -\frac{t^3}{3}, V_2(t) = \frac{2t^5}{15}, V_3(t) = -\frac{17t^7}{315}$$

One can verify that perturbation method also gives the same solution

Problem from wave propagation in rods with variable Young's modulus

(Bhattacharya and Bera: Applied Mathematics Letters (17), 2004)

Consider elastic bar of length *l*, cross section *A*, material density ρ and elastic modulus *E*. Assume that E varies with position.

The governing equation is

$$\frac{\partial}{\partial x}\left[AE \frac{\partial u}{\partial x} \right] = \rho A \frac{\partial^2 u}{\partial x^2},$$

with initial and boundary conditions

$$u(x,0) = 0 = \frac{\partial u(x,0)}{\partial t},$$

$$u(l,t) = kH(t),$$

$$u(0,t) = 0,$$

K is a constant and H(t) is the Heaviside function.

Write $$E = E(x) = E_0 + \varepsilon E_1(x)$$

Taking Laplace transform in time we can arrange it as

$$\frac{d^2U}{dx^2} + (1 - \frac{\varepsilon}{E_0}E_1)\frac{\varepsilon}{E_0}\frac{dE_1}{dx}\frac{dU}{dx} - \frac{s^2}{c^2}(1 - \frac{\varepsilon}{E_0}E_1)U = 0$$

$$U(0) = 0, \quad and \qquad U(l) = \frac{k}{s}$$

To apply Adomian method we put

$$L_0(U) = \frac{d^2U}{dx^2}$$

$$N_0(U) = a_1\frac{dU}{dx} - \frac{k^2}{c^2}(1 - a_1x)$$

$$a_1 = \frac{\varepsilon}{E_0}\frac{E_1}{l}$$

So that

$$L_0U = -N_0U$$

$$U_{n+1} = -L_0^{-1}\left[N_0U_n\right]$$

After some effort one can write the solution in the transformed plane as

$$U = \frac{ck}{ls^2}\sinh(\frac{sx}{c}) - \frac{a_1 x^2}{4}(\frac{k}{sl})\cosh(\frac{sx}{c}) - \frac{a_1 x}{4}(\frac{ck}{s^2 l})\sinh(\frac{sx}{c}) - - - -$$

Perturbation Method

If we apply perturbation Procedure on the transformed governing equation
The unperturbed problem corresponds to

$$\frac{d^2 U}{dx^2} - \frac{s^2}{c^2} U = 0$$

$$U(0) = 0, U(l) = \frac{k}{s}$$

$$U_0 = \frac{k}{s}(\csc h \frac{sl}{c}) \sinh \frac{s}{c} x$$

This gives the same leading term if expansion of *csch sl/c* is used.

Remarks

- The perturbation method seems to give the same result as Adomian method.
- The linear part of the operator in the Adomian method is the unperturbed (linearized) operator.
- The perturbation method provides some sense of smallness and the limitations involved which helps to select the unperturbed and perturbed part in the problem
- The claim that Adomian method yields fast convergence needs to be studied.

Liao's Homotopy Analysis Method

Claims

- The method is valid for even strongly nonlinear problems
- No need to identify a small / large parameter
- The convergence rate and region of convergence can be adjusted
- Different base functions can be used to suit the problem

Description of Method

Consider the problem (linear or non-linear)

$$M\left[u(\underline{r},t)\right]=0\cdots\cdots(1)$$

M is a nonlinear or linear operator, *u(r,t)* is an unknown function

$$Let\ u_0(\underline{r},t)\ denote\ an\ initial\ guess, h\neq 0$$
$$an\ auxiliary\ parameter, H(\underline{r},t)\neq 0\ an\ auxiliary\ function$$
$$and\ L\ an\ auxiliary\ linear\ operator$$

$$L\left[f(\underline{r},t)\right]=0 \quad when\ f(\underline{r},t)=0$$

Then, using the parameter q we define the homotopy

$$H\left[\phi(\underline{r},t,q);u_0(\underline{r},t),H(r,t),h,q\right]=$$
$$(1-q)\left\{L\left[\phi(\underline{r},t;q)-u_0(\underline{r},t\right]\right\}-qhH(\underline{r},t)M\left[\phi(\underline{r},t;q)\right]\cdots\cdots(2)$$

Enforcing the homotopy to be zero, we have the zero-th order deformation equation.

$$H\left[\phi(\underline{r},t,q);u_0(\underline{r},t),H(\underline{r},t),h,q\right]=0$$

Giving us

$$(1-q)\{L[\phi(\underline{r},t;q)-u_0(\underline{r},t]\}=qhH(\underline{r},t)M\left[\phi(\underline{r},t;q)\right],$$

$\phi(\underline{r},t)$ depends upon initial guess, the auxiliary parameter h, auxiliary linear operator *L* and embedding parameter q which lies in [0,1]. For q =0 The zero-th order deformation equation becomes

$$L\left[\phi(\underline{r},t)-u_0(\underline{r},t)\right]=0 \quad \text{or} \quad \phi(\underline{r},t)=u_0(\underline{r},t)$$

For q=1,

$$M\left[\phi(\underline{r},t;1)\right]=0 \quad \textit{which is the same as given equation if}$$
$$\phi(\underline{r},t)=u(\underline{r},t)$$

Thus as the embedding parameter changes from 0 to 1, $\phi(\underline{r},t)$ evolves from the initial guess to the exact solution. That is why the word deformation has been used in the context above.

Higher order

Define the m-th order derivatives

$$u_0^{[m]}(\underline{r},t) = \left[\frac{\partial^m \phi(\underline{r},t;q)}{\partial q^m}\right]_{q=0}$$

Expanding by Taylor's series and using the result on the left

$$\phi(\underline{r},t;q) = u_0(\underline{r},t) + \sum_{m=1}^{\infty} u_m(\underline{r},t) q^m$$

Implementation

Let us re-visit the simple example

$$V'(t) + V^2(t) = 1, \quad t \geq 0$$
$$V(0) = 0$$

The initial guess which satisfies the DE and the initial condition is

$$V_0(0) = 0$$

Let q be the embedding parameter. We choose the auxiliary linear operator

$$L[\phi(t;q)] = \alpha_1(t)\frac{\partial \phi(t;q)}{\partial t} + \alpha_2 \phi(t;q)$$

Where ♋’s are to be determined. Keeping in mind the given problem we can define the nonlinear operator

$$M[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t} + \phi^2(t;q) - 1$$

With all the notations as described before we define a family of equations

$$(1-q)L[\phi(t;q)-V_0(t)] = hqH(t)M[\phi((t;q)],$$
$$\phi(0;q)=0$$

In most cases choice of auxiliary parameter h and auxiliary function H(t) as well as the initial guess provides a flexibility that puts this method at an advantage as compared with other methods (Liao)
When q=0, we get

$$L[\phi(t;0)-V_0(t)] = 0, \quad \phi(0;0) = 0$$

The solution is the initial guess. q=1 gives $\phi(t;1) = V(t)$

As described in the method above

$$V_m(t) = \frac{1}{m!}\left[\frac{\partial^m \phi(t;q)}{\partial q^m}\right]_{q=o}$$

leading to

$$\phi(t;q) = V_0(t) + \sum_{m=1}^{\infty} V_m(t) q^m$$

Determination of V- functions

Consider $$(1-q)L\left[\phi(t;q)-V_0(t)\right]=hqH(t)M\left[\phi((t;q)\right],$$

$$\phi(0;q)=0$$

Differentiate this equation m times with respect to q and then put q=0, we obtain the so-called mth-deformation equation

$$L\left[V_m(t)-\chi_m V_{m-1}(t)\right]=hH(t)R_m(V_0--V_{m-1}),$$

$$V_m(0)=0$$

$$\chi_m = \begin{matrix} 0, & m\le 1 \\ 1 & m>1 \end{matrix}$$

Mathematica or some other software can be used to solve above.linear first order differential equation.

Some Recent work

- S. Liao: An explicit analytic solution to the Thomas-Fermi equation, Applied Mathematics and Computation, 144, 2-3, 2003.
- S. Liao: On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computation, 147, 2-3, 2004.

Concluding Remarks

- The perturbation method provides a measure of smallness which is inherent in the physics of the problem. This is not necessarily a drawback. It also helps to relate the solution of the perturbed problem to that of a perturbed problem
- The Adomian method and Homotopy method can well be applied if one has experience in perturbation to choose good first approximations
- The literature has some examples where an application of such a method without taking into account physics of the problem has led to un-realistic solutions of nonlinear problems
- One may take up a comparative study to study the rates of convergence in all three methods