Seismic Waves and Earthquakes – A Mathematical Overview

The destruction caused by earthquakes is caused by the seismic waves propagating inside and in particular, near the surface of earth. We describe the mathematical model and some mathematical techniques to deal with such waves in some interesting situations. As it turns out, the variation in earth structure plays a role in the intensity of earthquakes. It is therefore of great interest to identify the variations in earth geometry using the observed seismic signals. This constitutes the inverse seismic problem which shall be described and some recent results presented.

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Seismic Waves in Earth

A seismogram is a graph showing vibration or shaking of ground by recording the seismic waves. Seismic waves are propagating vibrations that carry energy from the source of the shaking outward in all directions.

The are many different seismic waves, but all are of basically four types:
An earthquake radiates P and S waves in all directions and the interaction of the P and S waves with Earth's surface and shallow structure produces surface waves.

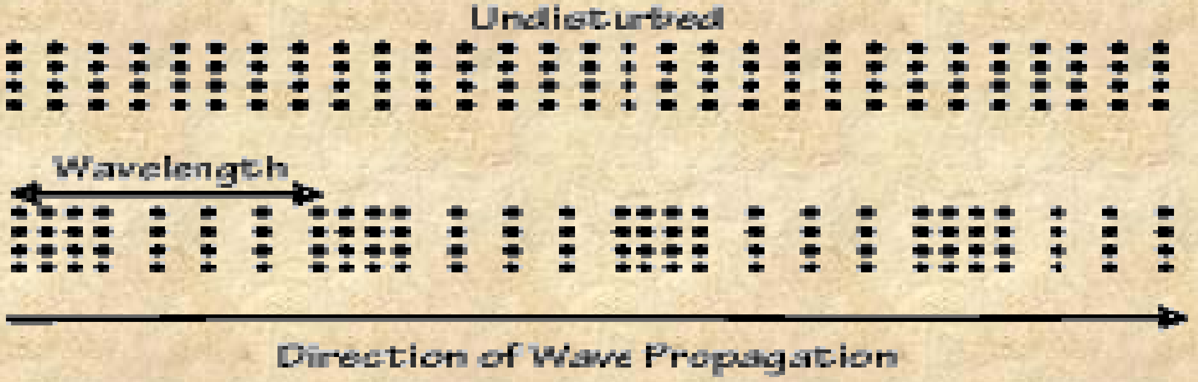
- Compress ional or P waves (for primary)
- Transverse or S waves (for secondary)
- Love waves
- Rayleigh waves

Compressional or P-Waves

P-waves are the first waves to arrive on a complete record of ground shaking because they travel the fastest (their name derives from this fact - P is an abbreviation for primary, first wave to arrive). They typically travel at speeds between ~1 and ~14 km/sec. The slower values corresponds to a P-wave traveling in water, the higher number represents the P-wave speed near the base of Earth's mantle.

The velocity of a P-wave depends on the elastic properties and density of a material:

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

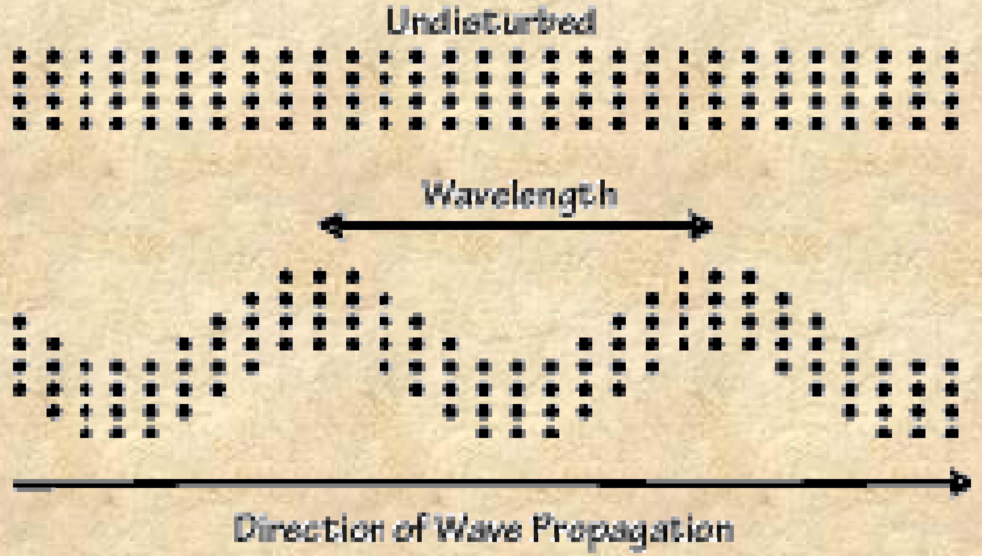


Shear or S- Waves

- Secondary, or S waves, travel slower than P waves
- These are also called "shear" waves
- S-waves are transverse waves because they vibrate the ground in a the direction "transverse", or perpendicular, to the direction that the wave is traveling.

$$velocity \quad \beta \quad is \quad given \quad by$$

$$\beta = \sqrt{\frac{\mu}{\rho}}$$



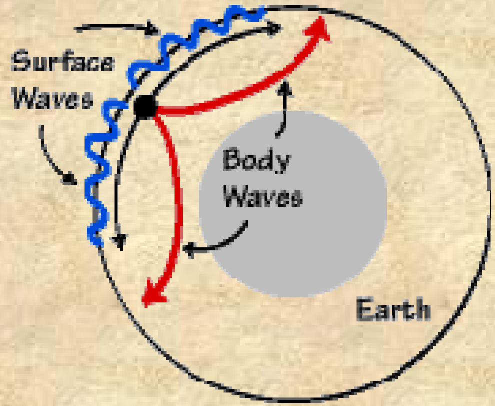
Surface Waves

An earthquake radiates P and S waves in all directions and the interaction of the P and S waves with Earth's surface and shallow structure produces surface waves. The amplitude of particle motion in such waves decreases with depth of the earth. The same type of waves can also propagate at the interface of two solid/solid or liquid/solid layers.

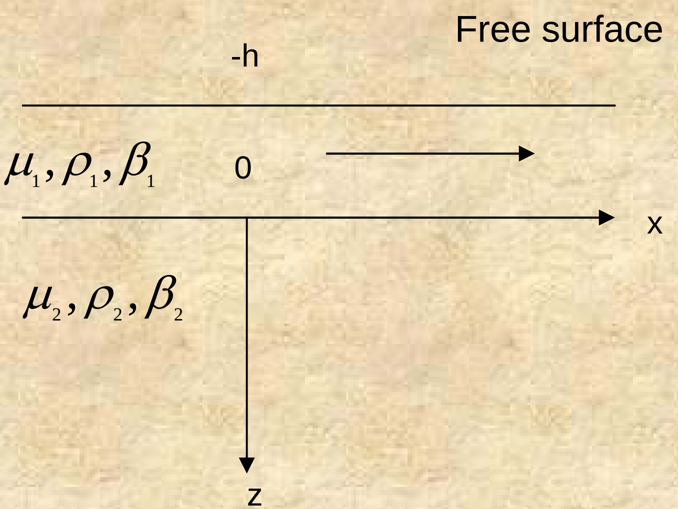
These are basically of two type.

- Love Waves: These are horizontally polarized shear (SH-) waves. Love wave travel in lateral layer overlying half space model of earth –h or in a wave-guide type layer inside earth.

- Rayleigh Waves: They are coupled vertically polarized shear (SV-) and P- waves.



Love Waves: Geometry and Governing Equations



$$\mu \Delta v + \rho f = \rho v_{tt},$$

μ *rigidity*

ρ *density*

f *body force – usually zero*

A time harmonic, uni-directional surface wave solution is of the form

$$v = Ae^{-bz} \exp\{ik(x - ct)\}$$

Which leads to dispersion relation

$$\tan\left\{\left(\frac{c^2}{\beta_1^2} - 1\right)^2 kh\right\} = \frac{\mu_2}{\mu_1}\frac{\left[1 - (c/\beta_2)^2\right]^{1/2}}{\left[1 - (c/\beta_1)^2\right]^{1/2}}$$

Scattering of Love waves in inhomogeneous medium

- The Love wave in a medium with variable density/rigidity is of interest due to applications in geophysical and earthquake engineering problems.
- Hudson (1962), Ghosh (1970), Chattopadhyay, Pal and Chakarborty (1984) considered Love waves traveling in a heterogeneous medium with or without a source.
- We have considered Love wave propagation problem in medium with variation in the elastic properties of the half space (Boll. Di Geofisica, 1990),
- Wave in an inhomogeneous layer trapped between two half space (J. Physics of Earth, 1990), Love waves excited by a source in an inhomogeneous layer (PUJM, 1991), and Love waves propagating in a layered half space with stochastic (*randomly varying*) properties (IL Nuovo Cimento C, 2004.
- The method is based upon integral transforms, perturbation and Green's function method.

Description of Method

In the inhomogeneous part of the medium (half space or/and overlying layer), the rigidity or/and density is assumed to vary as

$$\mu = \mu_0 + \varepsilon\mu_1, \rho = \rho_0 + \varepsilon\rho_1$$

The governing equation can be then written as

$$Lv = \nabla^2 v + k_0{}^2 v = \varepsilon f[v(x), \mu_1, \rho_1]$$

If $g(x;\xi)$ is the Green function satisfying the equation

$$Lg = \delta(x - \xi)$$

The solution to our problem is then written as

$$v(x) = \varepsilon \int_{\Omega} f(v(\xi), \mu_1, \rho_1) g(x;\xi) d\xi.$$

Transmission Problems: Spectral Representation Method

- The problem of transmission of seismic waves from a medium to another in which both media have different elastic properties also arises in practical situations.
- The spectral representation is sought on both sides using orthonormalised eigenfunctions
- In some cases the eigenfunctions are not proper in the sense that they are not square integrable but belong to a larger class of functions – these correspond to continuous spectrum and arise in infinite depth problems
- The spectrum plays an important role in the study of transmission and diffraction problem

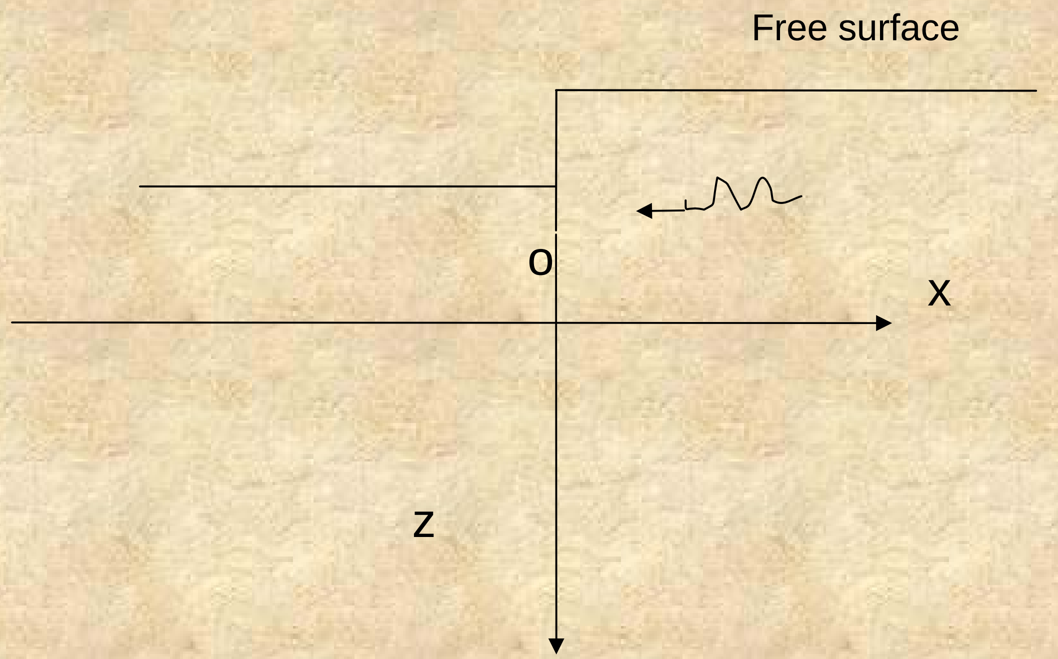
History

Kazi, M.H. (1976) studied spectral representation of Love waves by taking into account the contributions of the continuous spectrum. Kazi (1979) applied this to discuss transmission across a surface step. Niazy and Kazi (1980) Applied it to welded quarter spaces problem. Madja, Chin and Followill (1992)developed the perturbation theory based upon Kazi's spectral results.

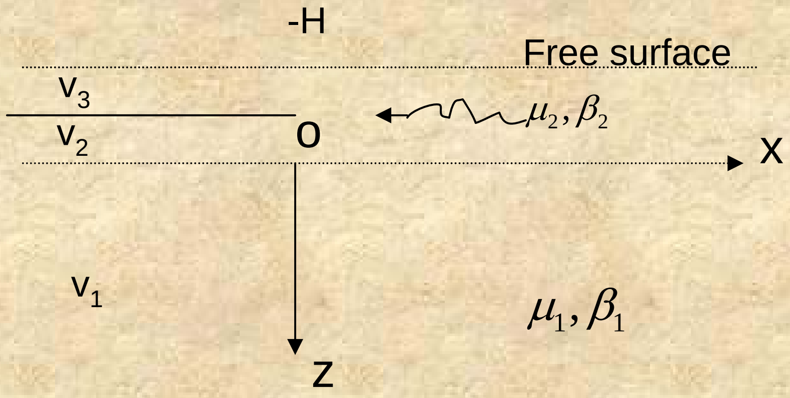
Our Result: Zaman et al studied spectral representation of Love-type waves For waves in a layer buried deep in the earth. A scattering theory can be built using this spectral representation.

Diffraction by Plane Discontinuities

Sato, R. (1961, J. Phys. Earth) considered a two dimensional model of homogeneous half space, overlain by a homogeneous surface layer which undergoes an abrupt change in thickness. His solution is based upon the Wiener-Hopf method and involves solution of an infinite set of simultaneous linear equations.



Kazi. M.H. (1976, Bull. Seism. Soc. Am.) used the Wiener-Hopf method to Study diffraction of Love waves by weak (crack) or rigid plane discontinuities of semi-infinite extent, lying in a layer overlying a half-space.



Conclusions

- Love waves incident on the half planes are diffracted into Love modes propagating in the region above crack (rigid plane) and channel waves in the region below crack (rigid plane) and above interface with half space.
- Channel modes die out rapidly with distance from edge of discontinuity.
- The problem of Love waves past a weak half-plane is connected to Sato's problem but has the advantage that only a finite set of equation have to be solved.

Asghar and Zaman have studied the model in which the plane discontinuity of finite length has been considered. This involves diffraction by two edges instead of one, and asymptotic approximation of the Fourier inversion integral. Kazi's results can be recovered from these by letting the length of plane discontinuity go to infinity:

1) *Diffraction of Love waves by a finite rigid barrier*, Bull. Seism. Soc. Am., 1986.
2) *Diffraction of SH-waves by a finite crack in a layer overlying a half space*, Boll. Di Geofisica Teorica ed Applicata, 1987.

Further extensions of these results were studied by considering two diffracting planes In the upper layer, interface of two half spaces and crack in a single half space:

3) Diffraction of Love waves by two staggered perfectly weak half planes, Il Nuovo Cimento, 1989

4) Diffraction of SH waves at a mixed interface, Il Nuovo Cimento, 1995.

5) *Scalar field due to a source in a half space in the presence of a crack*, Boll. Geofisica Teorica Ed Applicata, 1997.

Wiener-Hopf Technique

- Wiener and Hopf presented a novel technique in 1931 for solving certain singular Integral equations.

- Copson (1946) showed that mixed boundary value problems arising in diffraction problems could be formulated as singular integral equations and hence could be solved using the Wiener-Hopf technique.

- The integral equation was transformed into a functional equation in the complex plane using integral transforms and then solved using analytic continuation and Liouville's theorem.

- Jones (1952) applied integral transforms directly to the boundary value problem and introduced a method of obtaining the functional equation of the Wiener-Hopf type.
- To obtain the Wiener-Hopf functional equation, either of the Fourier or Laplace transform could be used.

- Noble (1958) has given an extensive discussion of the Wiener-Hopf technique and a good account of numerous problems solvable by this technique.

- Recently, the Wiener-Hopf technique has been also applied to problems arising in inverse scattering.

Various Steps

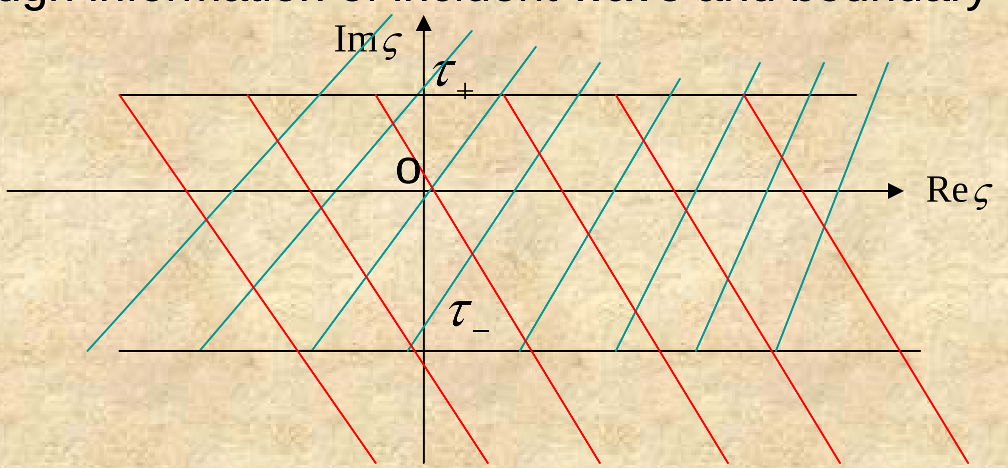
<u>Step 1</u>

Integral transforms are used to reduce the problem to that of solving the functional equation

$$F_+(\varsigma)G(\varsigma) = R_+(\varsigma) + S_-(\varsigma),\ \tau_- < \operatorname{Im}\varsigma < \tau_+$$

Where subscripts +, - refer to functions that are analytic in the upper half plane $\operatorname{Im}\varsigma > \tau_-$ (lower half plane $\operatorname{Im}\varsigma < \tau_+$).

$F_+(\varsigma)\ and\ S_-(\varsigma)$ are unknown functions while $G(\varsigma)\ and\ R_+(\varsigma)$ are known functions through information of incident wave and boundary conditions.



Step 2

Factorize $G(\varsigma)$ as $G(\varsigma) = G_+(\varsigma) G_-(\varsigma)$

Where $G_+(\varsigma)$ is analytic in the upper half plane $\operatorname{Im}\varsigma > \tau_-$ while $G_-(\varsigma)$ Is analytic in the lower half plane $\operatorname{Im}\varsigma < \tau_+$

The functions $G_+(\varsigma)$ *and* $G_-(\varsigma)$ are free of zeros in the respective half planes of their analyticity

We can re-arrange above equation as

$$F_+(\varsigma)G_+(\varsigma) = \frac{R_+(\varsigma)}{G_-(\varsigma)} + \frac{S_-(\varsigma)}{G_-(\varsigma)}, \quad \tau_- < \operatorname{Im}\varsigma < \tau_+,$$

We have now left hand side analytic in upper half plane, second term on LHS analytic in lower half plane but $\frac{R_+(\varsigma)}{G_-(\varsigma)}$ is a term which is mixed, i.e. analytic in neither half plane.

Step 3

We now use the additive decomposition

$$\frac{R_+(\varsigma)}{G_-(\varsigma)} = M_+(\varsigma) + M_-(\varsigma)$$

Where $M_\pm(\varsigma)$ are analytic in the upper (lower) half-plane.

Thus we now have

$$F_+(\varsigma)G_+(\varsigma) - M_+ = M_-(\varsigma) + \frac{S_-(\varsigma)}{G_-(\varsigma)}, \quad \tau_- < \operatorname{Im}\varsigma < \tau_+$$

The left side of this equation contains functions that are analytic in the upper half plane And right hand side is analytic in the lower half plane. Both sides are equal in the common strip $\tau_- < \operatorname{Im}\varsigma < \tau_+$.

By analytic continuation, both sides define an entire function $P(\varsigma)$.

This entire function can be determined by behavior of either side as $|\varsigma| \to \infty$.

Step 4

In most problems we find $P(\varsigma) \to 0 \quad as \quad |\varsigma| \to \infty.$

In such a case, we find the unknown functions given by

$$F_+(\varsigma) = \frac{M_+(\varsigma)}{G_+(\varsigma)}$$

and

$$S_-(\varsigma) = -M_-(\varsigma)G_-(\varsigma).$$

The factorizations needed in the above analysis can be achieved using theorems stated in Noble (1958) or Achenbach (1975).

The Wiener –Hopf technique, even today offers a useful tool to solve problems arising in a number of practical geophysical, material science, diffusion and inverse problems.

- Seismic waves, through which the earth-quake tremor travels, inter-act with *inhomogeneities* and discontinuities in the interior of earth.

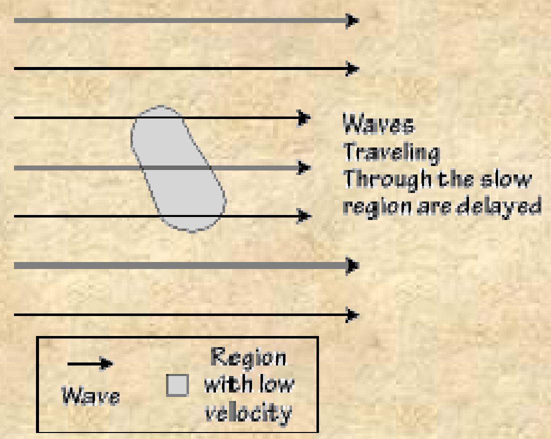
- It is therefore important to map interior of earth as accurately as possible.

- An accurate map of earth interior is also useful to exploration geologists.

- As a measure of change in earth structure, wave speed and variation in speed provides a good parameter to scientists.

- The idea is illustrated in the figure below. Waves are represented by arrows and are traveling from left to right. Those that travel through the slow region slow down, and hence are recorded later on a seismogram.

- The same ideas are used in medical CAT scan imaging of human bodies, but the observed quantity in a CAT scan is not a travel time, but the amount of x-ray absorption. Ultrasound imaging is identical to P-wave tomography, it's just that in seismology we don't have the choice of where are wave sources are located - we just exploit earthquakes.



Inverse Problem

- The inverse problem consists of finding the shape, size, location, depth and material properties of a body hidden deep inside the earth, by using the measured acoustic field caused by the body and measured on the surface of the earth.
- The objective of inverse problem can be defined as the determination of one or more parameters in the governing equation of some process.
- It is hoped that it will be possible to solve for those parameters using the observations of the scattered waves.
- The wave equation is transformed into Helmholtz equation.
- The Green function of the problem enables us to reduce the problem into an integral equation which can be solved for the unknown parameter/ velocity profile.
- One needs to solve a direct problem to find the Green function.

The Forward Scattering Problem in 1D

We assume that the "Field" is governed by this scalar *Helmholtz equation* with damping at a point x_s

$$\pounds u(x, x_s, \omega) = \frac{d^2 u(x, x_s, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma(x)}{\upsilon^2(x)}\right] u(x, x_s, \omega) = -\delta(x - x_s)$$

together with the radiation condition

$$\frac{du}{dx} \mp i \frac{\omega}{v(x)} u \to 0, \qquad as \qquad x \to \pm \infty$$

We introduce a notation that allows *variations* in damping and sound speed to have following form.

$$v(x) = v_0(x) + \delta v(x)$$

leading to

$$\frac{1}{v^2(x)} \approx \frac{1}{v_0^2(x)}[1 + \alpha(x)]$$

$$\gamma(x) = \gamma_0(x) + \delta\gamma(x)$$

The total field $u(x, x_s, \omega)$ is decomposed into *incident* and *scattered* field as

$$u(x, x_s, \omega) = u_S(x, x_s, \omega) + u_I(x, x_s, \omega)$$

Substituting this into the Helmholtz equation, comparing like terms and using the Born approximation leads to equations for incident field and scattered field.

$$£_0 u_I(x, x_s, \omega) = \frac{d^2 u_I(x, x_s, \omega)}{dx^2} + \left[\frac{\omega^2 + i\omega\gamma_0(x)}{\upsilon_0^2(x)}\right] u_I = -\delta(x - x_s),$$

$$£_0 u_S(x, x_S, \omega) = [-i\omega\delta\alpha(x) - \omega^2\alpha(x) - i\omega\gamma_0(x)\alpha(x)] \frac{u_I(x, x_S, \omega)}{v^2{}_0(x)}$$

First we assume that the source and receiver are placed at same point $x_S = x_g = 0.\ Choose\ \ v_0(x) = v_0\ \ and\ \ \gamma_0(x) = \gamma_0$

Last equation is solved using *Green's function* which can be found to be

$$g(x,0,\omega) = -\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})}.\exp(i\omega(1+\frac{i\gamma_0}{2\omega})x/v_0),$$

$$u_I(x,0,\omega) = g(x,0,\omega)$$

using this we get

$$u_S(0,\omega) = \int_0^\infty [\frac{-i\delta\gamma(x)}{4\omega(1+\frac{i\gamma_0}{2\omega})^2} - \frac{\alpha(x)}{4(1+\frac{i\gamma_0}{2\omega})^2} - \frac{i\gamma_0(x)\alpha(x)}{4\omega(1+\frac{i\gamma_0}{2\omega})^2}].e^{2i\omega(1+\frac{i\gamma_0}{2\omega})x/v_0}dx,$$

retaining only the leading order term in ω**, yields**

$$u_S(0,\omega) = -\int_0^\infty \frac{\alpha(x)}{4(1+\frac{i\gamma_0}{2\omega})^2}.e^{2i\omega(1+\frac{i\gamma_0}{2\omega})x/v_0}dx,$$

Inverse Scattering integral equation

$$\alpha(x) = \frac{-4e^{-\gamma_0 x/v_0}}{\pi v_0} \int_{-\infty}^{\infty} (1+\frac{i\gamma_0}{2\omega})^2 u_S(0,\omega).e^{-2i\omega x/v_0} d\omega.$$

For $\gamma_0 = 0$, it gives results which agree with those of the Bleistein model.

Comments

- We have formally reduced the inverse problem to the problem of solving the integral equation for $\alpha(x)$ using the observed data $u_S(0,\omega)$.
- The zero-offset constant background inversion formula derived here is the the first step to more involved inverse problems.

Some Inverse problems

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Further Work

- Investigation of inverse problems in Higher dimensions

- Determination of surface inhomogeneities or velocity profile inversion for surface
- waves

- Use of vector wave equation in inverse problems such as velocity inversion for
- Rayleigh waves.

- Inversion problems in mixed boundary values.

Thank You

Any questions or comments?