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Inverse scattering in multilayer inverse problem in the presence of damping

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Abstract

An inverse problem arising from recovery of wavespeed for a one-dimensional problem in a medium with constant background wavespeed in the presence of damping is discussed. Our method is based upon Born's approximation and the assumption that wavespeed and damping are well approximated by the background speed plus a perturbation term. An approximate form of Green's function for seismic data is used to derive the inversion formula. The procedure is then implemented on a medium which has two layers over which the wavespeed changes due to change in the physical properties.

Article Outline

1. Introduction

The inverse problem play an important role in mapping the interior of earth, geological prospecting and gaining information about inaccessible parts of different bodies. One or more signals are introduced near the surface of the body in a region of interest and responses from irregularities or inclusions in the interior are recorded. This received scattered profile is then used to recover the variations from a background medium. In case of seismic inversion, a homogenous elastic model with constant density and elastic parameters provides an important benchmark.

Under the assumption of constant density, Claerbout [2] presented an approximation method to the inverse problem for velocity inversion. Gerver [5] demonstrated that the velocity of propagation can be determined from observations at one point. Cohen and Bleistein [3], Cohen and Hagin [4] among others used Born's approximation [6] and high frequency assumptions to study the inverse problem in one and higher dimensions. The linearization used in the inversion procedure is often the Born's approximation. For this approximation, it is assumed that the variation in the physical parameters within an inhomogeneity is a small perturbation of the parameters of the background medium. An excellent account of the seismic inversion based upon these techniques can be found in Bleistein et al. [1]. Zaman and Masood [9] have introduced the damping effects with constant density in the model through damped wave equation [7]. The damping effects may be present in the medium due to moisture or other inhomogeneities. This model has been further used by Zaman et al. [10] to study recovery of damping parameter, wavespeed and bulk modulus under the high frequency assumption.

In this paper we present a one-dimensional model in a medium with damping and derive an inversion formula for variation in the velocity and the damping parameter of the medium. It is assumed that variation in the velocity and damping parameter is described by small perturbation terms so that Born's approximation can be used to linearize the inversion formula. The procedure is then applied to a multi layer medium in which the velocity and damping have two step changes from the background medium.

2. The one-dimensional inverse problem

The formulation of the forward scattering problem will be conducted in the frequency domain for some observable parameter, $u(x, x_s, \omega)$, called the "field". Here, x represents the general field, while x_s represents the location of the source, and ω represents frequency. The field may represent plane acoustic pressure waves (propagating parallel to the x-axis) in a two or three-dimensional medium, the transverse displacement of a string in one dimension, or some other equally appropriate parameter that may be represented as a one-dimensional wave. The only important condition is that propagating of $u(x, x_s, \omega)$ be governed by the scalar Helmholtz equation with damping.

$$\pounds u(x,x_s,\omega) = \frac{\mathrm{d}^2 u(x,x_s,\omega)}{\mathrm{d}x^2} + \left[\frac{\omega^2 + \mathrm{i}\omega\gamma(x)}{v^2(x)}\right] u(x,x_s,\omega) = -\delta(x-x_s), \tag{1}$$

together with the radiation condition

$$\frac{\mathrm{d}u}{\mathrm{d}x} \mp \mathrm{i}\frac{\omega}{v(x)} \to 0 \quad \text{as } x \to \pm\infty \tag{2}$$

with damping $\gamma(x)$ and source placed at point x_s. It is assumed that the source point is to the left of the region where $v(x)$ and $\gamma(x)$ are unknown. We introduce a notation that allows variation in damping and sound speed to have parallel form as in equation (3). Therefore, introduce the reference velocity v_0, and a perturbation α in the form

$$\begin{aligned}\frac{1}{v^2(x)} &\approx \frac{1}{v_0^2(x)}[1+\alpha(x)],\\ \gamma(x) &= \gamma_0(x) + \delta\gamma(x),\end{aligned} \tag{3}$$

rewriting (1) using the perturbation representation (3) yields an equivalent Helmholtz equation

$$\begin{aligned}£_0 u(x,x_s,\omega) &= \frac{\mathrm{d}^2 u(x,x_s,\omega)}{\mathrm{d}x^2} + \left[\frac{\omega^2 + \mathrm{i}\omega\gamma_0(x)}{v_0^2(x)}\right]u\\ &= -\delta(x-x_s) + \left[-\mathrm{i}\omega\delta\gamma(x) - \omega^2\alpha(x) - \mathrm{i}\omega\gamma_0(x)\alpha(x)\right]\frac{u(x,x_s,\omega)}{v_0^2(x)}.\end{aligned} \tag{4}$$

Here, the term involving $\alpha(x)$ has moved to the right side of the equation. Eq. (4) is posed in terms of the total field $u(x, x_s, \omega)$ generated by the impulsive source $-\delta(x - x_s)$ plus the more complicated "scattering source" represented by the term on the far right in (4). The scattered waves generated by this new "source" have interacted with regions at greater depth than x_s and x_g and contain information about the wavespeed profile at these greater depths. The total field $u(x, x_s, \omega)$ can be separated into the incident part $u_\mathrm{I}(x, x_s, \omega)$ in the absence of the perturbations and $u_\mathrm{S}(x, x_s, \omega)$ in the presence of perturbations. Thus set

$$u(x,x_s,\omega)=u_\mathrm{S}(x,x_s,\omega)+u_\mathrm{I}(x,x_s,\omega). \tag{5}$$

Substitute (5) in (4) and, we reach at the following equations:

$$£_0 u_\mathrm{I}(x,x_s,\omega) = \frac{\mathrm{d}^2 u_\mathrm{I}(x,x_s,\omega)}{\mathrm{d}x^2} + \left[\frac{\omega^2 + \mathrm{i}\omega\gamma_0(x)}{v_0^2(x)}\right]u_\mathrm{I} = -\delta(x-x_s), \tag{6}$$

$$£_0 u_\mathrm{S}(x,x_s,\omega) = \left[-\mathrm{i}\omega\delta\gamma(x) - \omega^2\alpha(x) - \mathrm{i}\omega\gamma_0(x)\alpha(x)\right]\frac{[u_\mathrm{I}(x,x_s,\omega) + u_\mathrm{S}(x,x_s,\omega)]}{v_0^2(x)}. \tag{7}$$

We now construct a Green's function representation of (7)

$$\frac{\mathrm{d}^2 g(x,x_g,\omega)}{\mathrm{d}x^2} + \left[\frac{\omega^2 + \mathrm{i}\omega\gamma_0(x)}{v_0^2(x)}\right]g(x,x_g,\omega) = -\delta(x-x_g). \tag{8}$$

Consider the case where the source and receiver are located at same place (for simplicity we choose $x_s = x_g \equiv 0$, $v_0(x) = v_0$ and $\gamma_0(x) = \gamma_0$). Let

$$u_\mathrm{I}(x,0,\omega)=g(x,0,\omega).$$

This is the "zero-offset problem". Following Wu [8] we assume

$$\sqrt{\frac{\omega^2 + \mathrm{i}\omega\gamma_0}{v_0^2}} \approx \frac{\omega}{v_0}\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right). \tag{9}$$

The Green's function may then be written as

$$g(x, 0, \omega) \approx -\frac{v_0}{2\mathrm{i}\omega\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)} \exp\left(\mathrm{i}\omega\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)x/v_0\right). \tag{10}$$

The solution of (7), after Born's approximation [1, Chapter 2], written in terms of Green's function can be written as

$$u_S(0, \omega) = -\int_0^\infty \left[-\mathrm{i}\omega\delta\gamma(x) - \omega^2\alpha(x) - \mathrm{i}\omega\gamma_0(x)\alpha(x)\right] \frac{g^2(x, 0, \omega)}{v_0^2(x)}\, \mathrm{d}x. \tag{11}$$

Now using the Green's function representations (10) in (11),

$$u_S(0, \omega) = \int_0^\infty \left[\frac{-\mathrm{i}\delta\gamma(x)}{4\omega\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)^2} - \frac{\alpha(x)}{4\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)^2} - \frac{\mathrm{i}\gamma_0(x)\alpha(x)}{4\omega\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)^2}\right] \mathrm{e}^{2\mathrm{i}\omega\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)x/v_0}\, \mathrm{d}x \tag{12}$$

and retaining only the leading order terms in ω, we have

$$u_S(0, \omega) = -\int_0^\infty \frac{\alpha(x)}{4\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)^2} \mathrm{e}^{2\mathrm{i}\omega\left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)x/v_0}\, \mathrm{d}x. \tag{13}$$

Since $\alpha(x) = 0$ for $x < 0$, (13) is Fourier type integral because lower limit can be extended to $-\infty$. This can be treated as a Fourier transform of $\alpha(x)$ and inversion can be performed. The result is

$$\alpha(x) = \frac{-4\mathrm{e}^{-\gamma_0 x/v_0}}{\pi v_0} \int_{-\infty}^\infty \left(1 + \frac{\mathrm{i}\gamma_0}{2\omega}\right)^2 u_S(0, \omega)\mathrm{e}^{-2\mathrm{i}\omega x/v_0}\, \mathrm{d}\omega. \tag{14}$$

As simple check on this result, note that when $\gamma_0 = 0$, this results reduces to the constant-background inversion formula [1].

3. Inversion formula for data gathered in multilayer model

We will apply our constant-background inversion formula (14) to data gathered in the two-layer model. Let $u(x, \omega)$ be a solution of the problem (1) with radiation condition. Here

$$v(x) = \begin{cases} v_0, & \text{for } x < h_1, \\ v_1, & \text{for } h_1 < x < h_2, \\ v_2, & \text{for } h_2 < x, \end{cases} \qquad \gamma(x) = \begin{cases} \gamma_0, & \text{for } x < h_1, \\ \gamma_1, & \text{for } h_1 < x < h_2, \\ \gamma_2, & \text{for } h_2 < x. \end{cases} \tag{15}$$

We begin to solve this problem by writing down fairly general solutions in each of the three regions, with constant to be determined by interface and radiation conditions. The solution satisfying the radiation condition can be found to be

$$
\begin{aligned}
u &= -\frac{v_0}{2\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)}\left[\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)x/v_0}+A_1\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)(2h_1-x)/v_0}\right], \quad x<h \\
u &= -\frac{v_0}{2\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)}\left\{A_2\mathrm{e}^{\mathrm{i}\omega\left[\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)h_1/v_0+\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)(x-h_1)/v_1\right]}\right. \\
&\quad \left.+A_3\mathrm{e}^{\mathrm{i}\omega\left[\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)h_1/v_0+\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)(h_1-x)/v_1\right]}\right\}, \quad h_1<x<h_2, \\
u &= -\frac{v_0}{2\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)}A_4\left[\mathrm{e}^{\mathrm{i}\omega\left[\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)h_1/v_0+\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)(h_2-h_1)/v_1+\left(1+\frac{\mathrm{i}\gamma_2}{2\omega}\right)(x-h_2)/v_2\right]}\right], \quad h_2<x.
\end{aligned}
\tag{16}
$$

We require that u and its first derivative are continuous at $x = h_1$ and $x = h_2$. This requirement leads to the following system of equations:

$$A_1\text{-}A_2\text{-}A_3\text{=-}1,$$

$$-\frac{\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)A_1}{v_0}-\frac{\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_2}{v_1}+\frac{\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_3}{v_1}=-\frac{\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)}{v_0}, \tag{17}$$

$$\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_2\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)\tau/2v_1}+\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_3\mathrm{e}^{-\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)\tau/2v_1}-\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_4\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_2}{2\omega}\right)\tau/2v_1}=0,$$

$$\frac{\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_2\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)\tau/2v_1}}{v_1}-\frac{\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)A_3\mathrm{e}^{-\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)\tau/2v_1}}{v_1}-\frac{\left(1+\frac{\mathrm{i}\gamma_2}{2\omega}\right)A_2\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_2}{2\omega}\right)\tau/2v_1}}{v_2}=0,$$

where $\tau = 2(h_2 - h_1)$.

Solving for A_1, A_2, and A_3 yields

$$u_S(0,\omega)=-F(\omega)\frac{v_0}{2\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)}\left[1+\frac{R_1+R_2\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)\tau/v_1}}{1+R_1R_2\mathrm{e}^{\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)\tau/v_1}}\mathrm{e}^{2\mathrm{i}\omega\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)h_1/v_0}\right], \tag{18}$$

where

$$
\begin{aligned}
R_1 &= \frac{\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)v_1-\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)v_0}{\left(1+\frac{\mathrm{i}\gamma_0}{2\omega}\right)v_1+\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)v_0}, \\
R_2 &= \frac{\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)v_2-\left(1+\frac{\mathrm{i}\gamma_2}{2\omega}\right)v_1}{\left(1+\frac{\mathrm{i}\gamma_1}{2\omega}\right)v_2+\left(1+\frac{\mathrm{i}\gamma_2}{2\omega}\right)v_1}.
\end{aligned}
\tag{19}
$$

By expanding the denominator in (18) in a geometric series $\left(\left|R_1R_2\mathrm{e}^{\mathrm{i}\omega(1+\frac{\mathrm{i}\gamma_1}{2\omega})\tau/v_1}\right|<1\right)$, the response can be written as

(20)

$$u_S(0,\omega) = -F(\omega)\frac{v_0}{2i\omega(1+\frac{i\gamma_0}{2\omega})}\left\{1 + R_1 e^{2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)h_1/v_0} + R_2[1-R_1^2]\right.$$
$$\left.\times\sum_{n=2}^{\infty}[-R_1R_2]^{n-1}e^{in\omega\left(1+\frac{i\gamma_1}{2\omega}\right)\tau/v_1 + 2i\omega\left(1+\frac{i\gamma_0}{2\omega}\right)h_1/v_0}\right\}.$$

First, consider the case $F(\omega) = 1$. The Fourier inversion of each term in this series can be carried out to obtain the following expression giving the variation in velocity profile.

$$\alpha(x) = \frac{-4e^{-\gamma_0 x/v_0}}{\pi}\int_{-\infty}^{\infty}\frac{(1+\frac{i\gamma_0}{2\omega})}{2i\omega}\left\{R_1 e^{-\gamma_0 h_1/v_0}e^{2i\omega(h_1-x)/v_0} + R_2[1-R_1^2]\right.$$
$$\left.\times\sum_{n=2}^{\infty}[-R_1R_2]^{n-1}e^{-n\gamma_1(h_2-h_1)/v_1-\gamma_0 h_1/v_0}e^{2in\omega(h_2-h_1)/v_1+2i\omega(h_1-x)/v_0}\right\}d\omega. \tag{21}$$

If R_1 and R_2 are independent of ω, that is

$$R_1 = \frac{v_1 - v_0}{v_1 + v_0},\qquad R_2 = \frac{v_2 - v_1}{v_2 + v_1}, \tag{22}$$

then

$$\alpha(x) \approx \frac{-4e^{-\gamma_0 x/v_0}}{\pi}\int_{-\infty}^{\infty}\frac{1}{2i\omega}\left\{R_1 e^{-\gamma_0 h_1/v_0}e^{2i\omega(h_1-x)/v_0} + R_2[1-R_1^2]\right.$$
$$\left.\times\sum_{n=2}^{\infty}[-R_1R_2]^{n-1}e^{-2n\gamma_1(h_2-h_1)/v_1-\gamma_0 h_1/v_0}e^{2i\omega[n(h_2-h_1)/v_1+(h_1-x)]/v_0}\right\}d\omega. \tag{23}$$

We find that

$$\alpha(x) \approx 4\left\{[R_1 e^{-\gamma_0(h-x)/v_0}H(x-h)] + R_2[1-R_1^2]e^{-2\gamma_1(h_2-h_1)/v_1-\gamma_0(h-x)/v_0}H((h_2-h_1)v_0/v_1 + h_1 - x\right.$$
$$\left.+e^{-2n\gamma_1(h_2-h_1)/v_1-\gamma_0(h-x)/v_0}\sum_{n=2}^{\infty}[-R_1R_2]^{n-1}H(n(h_2-h_1)v_0/v_1 + h_1 - x)\right\}.$$

4. Numerical results and graphs

For two layers, the band-filter for the band

$$[-\omega_4,-\omega_3]\cup[-\omega_3,-\omega_2]\cup[-\omega_2,-\omega_1]\cup[\omega_1,\omega_2]\cup[\omega_2,\omega_3]\cup[\omega_3,\omega_4]$$

has the transform functions as in Fig. 1. The band limiting of $\alpha(x)$ can be obtained by the convolution theorem. Since we have Eq. (24). Let

$$x = v_0 t,\qquad h = v_0 t_0$$

and by convolution theorem

$$\psi(t)_{\text{bandlimited}} = \psi(t) * F(\omega), \tag{25}$$

where $F(\omega)$ is the filter which is symmetric and nonnegative in the ω-domain. The filter is given by the resulting equation

$$\text{filter} = \frac{\cos\omega_4 t - \cos\omega_3 t}{\pi\Delta_1 \omega t^2} + \frac{\cos\omega_2 t - \cos\omega_1 t}{\pi\Delta_2 \omega t^2}, \tag{26}$$

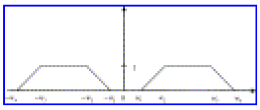
where

$$\Delta_1 = \omega_4 - \omega_3,$$
$$\Delta_2 = \omega_2 - \omega_1,$$

from Eq. (24), the band limiting of $\alpha(x)$ for two layers is

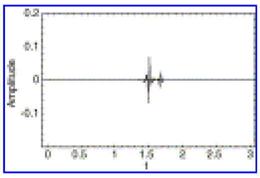
$$\psi(t)_{\text{bandlimited}} = -\frac{4}{\pi}\int_0^\infty \{R_1 e^{-\gamma_0(t_0-\tau)/v_0} H([\tau - t_0]v_0) + R_2[1-R_1^2]e^{-2\gamma_1(h_2-h_1)/v_1-\gamma_0(h-x)/v_0} H$$
$$\times((h_2-h_1)v_0/v_1 + h_1 - x)\}\left[\frac{\cos\omega_4(t-\tau) - \cos\omega_3(t-\tau)}{\pi\Delta_1\omega(t-\tau)^2} + \frac{\cos\omega_2(t-\tau) - \text{co}}{\pi\Delta_2\omega(t-}\right.$$

The first term in this expression is just what can be obtained for the single layer, $-4R_1$, where R_1 is the reflection coefficient of the first boundary. For small perturbations $\alpha = O(\varepsilon)$, this term reproduces the step at $x = h_1$ to all orders in ε. The second term produces a step at $x = h_1 + (h_2 - h_1)c_0/c_1$, instead of a step at $x = h_2$ as shown in Fig. 2 for frequencies of 10, 20, 50, and 60 Hz, respectively. The amplitude, $R_2[1 - R_1^2]$ is correct to order ε.



(6K)

Fig. 1. The band-limiting for the band $F(w)$ for four frequencies.



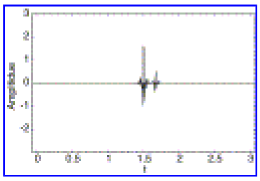
(10K)

Fig. 2. The bandwidth of the data is a trapezoid with corner frequencies of 10, 20, 50 and 60 Hz, respectively.

Let us now return to the band limited data, that is to say $F(\omega)$ is no longer identically equal to 1. We compute the reflectivity function $\beta(x)$, to the solution representation (20).

$$\begin{aligned}\beta(x) = &-\frac{4R_1 e^{-\gamma_0(h-x)/v_0}}{\pi v_0}\delta_B(x-h_1) + \frac{4R_1\gamma_0 e^{-\gamma_0(h-x)/v_0}}{v_0}H(x-h_1)\\ &-\frac{4R_2[1-R_1^2]e^{-2\gamma_1(h_2-h_1)/v_1-\gamma_0(h-x)/v_0}}{\pi v_0}\delta_B((h_2-h_1)v_0/v_1+h_1-x)\\ &+\frac{4R_2[1-R_1^2]\gamma_0 e^{-2\gamma_1(h_2-h_1)/v_1-\gamma_0(h-x)/v_0}}{\pi v_0}H((h_2-h_1)v_0/v_1+h_1-x)\end{aligned} \tag{28}$$

as the out put from the inversion formula as in Fig. 3.



(9K)

Fig. 3. Step data 10, 20, 50 and 60 Hz, respectively, (with derivative operator).

5. Conclusion

We solve the inverse problem of determination of velocity profile in a mutilayered medium by writing down fairly general solutions in the multilayered regions, with constants to be determined by interface and radiation conditions. We derive approximate solutions to the inverse problem of finding the velocity and damping from the observed wave-field. In order to evaluate the singular integral involved in the inversion formula, a band limiting filter for delta function is used. The results are presented graphically.

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