ANALYSIS AND FINITE ELEMENT APPROXIMATION OF A LADYZHENSKAYA MODEL FOR VISCOUS FLOW IN STREAMFUNCTION FORM [1]

Faisal A. Fairag

Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals, KFUPM Box 1533, Dhahran 31261, Saudi Arabia.

Abstract

In this paper we consider a model for the motion of incompressible viscous flows proposed by Ladyzhenskaya. The Ladyzhenskaya model is written in terms of the velocity and pressure while the studied model is written in terms of the streamfunction only. We derived the streamfunction equation of the Ladyzhenskaya model and present a weak formulation and show that this formulation is equivalent to the velocity-pressure formulation. We also present some existence and uniqueness results for the model. Finite element approximation procedures are presented. The discrete problem is proposed to be well posed and stable. Some error estimates are derived. We consider the 2D driven cavity flow problem and provide graphs which illustrate differences between the approximation procedure presented here and the approximation for the streamfunction form of the Navier-Stokes equations. Streamfunction contours are also displayed showing the main features of the flow.

Key words: Finite element method, Ladyzhenskaya model, streamfunction formulation.
1991 MSC: 65N30, 76M10, 78M10, 76D03, 35Q35.

1 Introduction

In [23–25], Ladyzhenskaya has proposed a model for the motion of ideal incompressible flow. An excellent piece of motivation why one consider this model

Email address: ffairag@kfupm.edu.sa (Faisal A. Fairag).
URL: http://faculty.kfupm.edu.sa/math/ffairag (Faisal A. Fairag).

[1] This work was supported by KFUPM under grant INT-278.

can be found in [10]. Du and Gunzburger mentioned several reasons to consider this model. They are modeling, mathematical, practical engineering and practical programming point of views. Ladyzhenskaya presented her model in velocity-pressure version. Further studies are made in [8–10,27].

In this paper, we study the streamfunction equation of Ladyzhenskaya model. The attractions of the streamfunction equation are that the incompressibility constraint is automatically satisfied, the pressure is not present in the weak form and there is only one scalar unknown to solve for. The purpose of this paper is to present and analyze a weak formulation for the streamfunction of the Ladyzhenskaya model and its discretization.

We first need to state the Ladyzhenskaya model in velocity-pressure form. Let Ω be a bounded, simply connected, polygonal domain in R^2 and $\vec{u}$ denotes the velocity field, p the pressure and $\vec{f}$ the body force. The Ladyzhenskaya equations for two dimensional incompressible fluid flow are

$$-\partial_x(\hat{A}(u)u_{1,x}) - \partial_y(\hat{A}(u)u_{1,y}) + u_1u_{1,x} + u_2u_{1,y} + p_x = f \quad \text{in} \quad \Omega, \tag{1}$$
$$-\partial_x(\hat{A}(u)u_{2,x}) - \partial_y(\hat{A}(u)u_{2,y}) + u_1u_{2,x} + u_2u_{2,y} + p_y = f \quad \text{in} \quad \Omega, \tag{2}$$
$$u_x + u_y = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

with the homogeneous Dirichlet boundary conditions on u_1 and u_2, i.e.

$$u_1 = u_2 = 0 \quad \text{on} \quad \partial\Omega, \tag{4}$$

where in (2)

$$\hat{A}(\vec{u}) = \varepsilon_0 + \varepsilon_1 \mid \nabla\vec{u} \mid^{q-2} \quad \text{with} \quad q > 2, \tag{5}$$

and

$$\mid \nabla\vec{u} \mid = [u_{1,x}^2 + u_{1,y}^2 + u_{2,x}^2 + u_{2,y}^2]^{1/2}. \tag{6}$$

We also assume that $\frac{1}{Re} = \varepsilon_0 > 0$ and $\varepsilon_1 > 0$ are constants. Note that if we set $\varepsilon_1 = 0$, equations (1-3) become the familiar Navier-Stokes equations.

Any divergence-free velocity field, $\vec{u}$, in $H_0^1(\Omega)$ has a streamfunction ψ defined by

$$\vec{\text{curl}}\ \psi = \vec{u}.$$

Moreover, ψ is uniquely determined up to a constant. Since $\frac{\partial\psi}{\partial\tau} = 0$ on $\partial\Omega$, where τ denotes the unit tangent to $\partial\Omega$, setting $\psi = 0$ on $\partial\Omega$ guarantees the uniqueness of the streamfunction.
Thus we have

$$-\partial_x(A(\psi)\psi_{xy}) - \partial_y(A(\psi)\psi_{yy}) + \psi_y\psi_{xy} - \psi_x\psi_{yy} + p_x = f_1 \quad \text{in} \quad \Omega, \tag{7}$$
$$-\partial_x(A(\psi)\psi_{xx}) - \partial_y(A(\psi)\psi_{xy}) + \psi_y\psi_{xx} - \psi_x\psi_{xy} + p_y = f_2 \quad \text{in} \quad \Omega, \tag{8}$$
$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{9}$$

where in (7) and (8), n represents the outward unit normal to Ω and $A(\psi)$ is defined by

$$A(\psi) = \varepsilon_0 + \varepsilon_1 \parallel \vec{\triangle}\psi \parallel^{q-2}, \tag{10}$$

and

$$\vec{\triangle}\psi = \vec{\text{grad}}\ (\vec{\text{grad}}\ \psi) = [\psi_{xx}, \psi_{xy}, \psi_{yx}, \psi_{yy}]^T,$$

and

$$\parallel \vec{\triangle}\psi \parallel = [\psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2]^{1/2}. \tag{11}$$

Taking the " curl " of (7) and (8) will eliminate the pressure p and yields the streamfunction equation of the Ladyzhenskaya equations

$$\begin{aligned} \partial_{xx}(A(\psi)\psi_{xx}) + 2\partial_{xy}(A(\psi)\psi_{xy}) + \partial_{yy}(A(\psi)\psi_{yy}) - \\ \psi_y \triangle \psi_x + \psi_x \triangle \psi_y = f_{2,x} - f_{1,y} \quad \text{in} \quad \Omega, \\ \psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{12}$$

Equation (12) is the particular equation we consider in this paper. We also state the streamfunction equation of the Navier-Stokes equations

$$\begin{aligned} \varepsilon_0 \triangle^2 \psi - \psi_y \triangle \psi_x + \psi_x \triangle \psi_y = f_{2,x} - f_{1,x} \quad \text{in} \quad \Omega, \\ \psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{13}$$

Note that if we set $\varepsilon_1 = 0$, equation (12) reduces to equation (13).

2 Notation And Function Spaces

We start by introducing some function spaces. First, let us define

$D(\Omega) := C_0^\infty(\Omega) =$ the space of (real-valued) smooth functions with compact support in the domain Ω,

$L^2(\Omega) :=$ the space of (real-valued) functions which are square integrable over Ω with respect to the Lebegue measure,

$L_0^2(\Omega) :=$ the space of functions in $L^2(\Omega)$ with mean zero.

We also define the Soboleve space

$$W^{m,p}(\Omega) := \{\phi \in L^p(\Omega) : \partial^\alpha \phi \in L^p(\Omega) \quad \forall \mid \alpha \mid \leq m\},$$

which is a Banach space for the norm

$$\parallel \phi \parallel_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \int_\Omega \mid \partial^\alpha \phi(x) \mid^p d\Omega \right)^{1/p} \qquad p < \infty.$$

The space $W^{m,p}(\Omega)$ is separable and reflexive. We also provide $W^{m,p}(\Omega)$ with the following seminorms

$$| \phi |_{m,p,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega | \partial^\alpha \phi(x) |^p \, d\Omega \right)^{1/p},$$

when $p = 2, W^{m,p}(\Omega)$ is usually denoted by $H^m(\Omega)$. $H^m(\Omega)$ is a Hilbert space for the scalar product

$$(\psi, \phi)_{m,\Omega} = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha \phi(x) \partial^\alpha \psi(x) d\Omega.$$

As $D(\Omega) \subset W^{2,q}(\Omega) \quad (q > 2)$ and $D(\Omega) \subset H^2(\Omega)$. We define

$$\begin{aligned} V &:= H_0^2(\Omega) := \text{ completion of } D(\Omega) \text{ in the } H^2\text{-norm}, \\ V_q &:= W_0^{2,q}(\Omega) := \text{ completion of } D(\Omega) \text{ in the } W^{V_q}\text{-norm}, \quad (q > 2). \end{aligned}$$

We also define the following spaces

$$\begin{aligned} \mathcal{G} &:= \{u \in [D(\Omega)]^2 : \text{div } u = 0\}, \\ \hat{V} &:= \text{completion of } \mathcal{G} \text{ in the } H^1\text{-norm}, \\ \hat{V}_q &:= \text{completion of } \mathcal{G} \text{ in the } W^{1,q}\text{-norm} \quad (q > 2). \end{aligned}$$

The space $\hat{V}$ is a Hilbert space with corresponding inner product and norm

$$\begin{aligned} (u, v) &:= \int_\Omega \nabla u : \nabla v \, d\Omega \quad \text{for } u, v \in \hat{V}, \\ \| u \|_{1,2} &:= \left[\int_\Omega | \nabla u |^2 \, d\Omega \right]^{1/2}. \end{aligned}$$

The space $\hat{V}_q$ is a reflexive Banach space, endowed with the following norm : for $u \in \hat{V}_q$,

$$\| u \|_{1,q} := \left[\int_\Omega | \nabla u |^q \, d\Omega \right]^{1/q}.$$

The spaces L^2 and V are Hilbert spaces with corresponding inner products and norms

$$\begin{aligned} (\psi, \phi) &:= \int_\Omega \psi \cdot \phi \, d\Omega \quad \text{for} \quad \psi, \phi \in L^2(\Omega), \\ \| \phi \|_{0,2} &:= (\phi, \phi)^{1/2}. \end{aligned}$$

Similarly,

$$< \psi, \phi > := \int_\Omega (\psi_{xx}\phi_{xx} + 2\psi_{xy}\phi_{xy} + \psi_{yy}\phi_{yy}) d\Omega \quad \text{for} \quad \psi, \phi \in V,$$
$$\| \phi \|_V := | \phi |_V := [\int_\Omega \| \vec{\triangle}\phi \|^2 \, d\Omega]^{1/2}.$$

Also, L^q and V^q are reflexive Banach spaces, with the following norms

$$\| \phi \|_{0,q} := \left[\int_\Omega | \phi |^q \, d\Omega \right]^{1/q} \qquad \text{for} \quad \phi \in L^q(\Omega),$$
$$\| \phi \|_{V_q} := \left[\int_\Omega \| \vec{\triangle}\phi \|^q \, d\Omega \right]^{1/q} \qquad \text{for} \quad \phi \in V^q.$$

Two applications of Green's formula show

$$| \phi |_V = | \triangle\phi |_{0,2} \quad \forall \phi \in V.$$

3 Weak Formulations

The weak form of (1-4) [see [9]] is:

$$\begin{array}{l} \text{For} \quad \vec{f} \in \hat{V}' \quad \text{given, Find} \quad \vec{u} \in \hat{V}_q \quad \text{satisfying} \\ (\hat{A}(\vec{u}) \bigtriangledown \vec{u}, \bigtriangledown \vec{v}) + \hat{b}(\vec{u}, \vec{u}, \vec{v}) = (\vec{f}, \vec{v}) \quad \forall \quad \vec{v} \in \hat{V}_q, \end{array} \tag{14}$$

where

$$\hat{b}(\vec{u}, \vec{v}, \vec{w}) := \int_\Omega ((\vec{u} \cdot \bigtriangledown)\vec{v}) \cdot \vec{w} \, d\Omega.$$

We can establish the weak form for the streamfunction equation of the Ladyzhenskaya equation by first multiplying equation (12) by a test function $\phi \in V_q$, and integrating over the domain Ω and then applying Green's formula twice to get

$$\begin{array}{l} \int_\Omega A(\psi)(\psi_{xx}\phi_{xx} + 2\psi_{xy}\phi_{xy} + \psi_{yy}\phi_{yy}) d\Omega \, + \\ \qquad \int_\Omega \triangle\psi(\psi_y\phi_x - \psi_x\phi_y) d\Omega = \int_\Omega (f_1\phi_y - f_2\phi_x) d\Omega, \end{array} \tag{15}$$

for all $\phi \in V_q$. Also, we can rewrite (15) as

$$\int_\Omega A(\psi)(\vec{\triangle}\psi \cdot \vec{\triangle}\phi) d\Omega + \int_\Omega \triangle\psi(\psi_y\phi_x - \psi_x\phi_y) d\Omega = \int_\Omega (f_1\phi_y - f_2\phi_x) d\Omega,$$

and we conclude that the weak form of the equation (12) is

$$\begin{array}{l} \text{Find} \quad \psi \in V_q \quad \text{such that, for all} \quad \phi \in V_q \\ a(\psi, \psi, \phi) + b(\psi, \psi, \phi) = (\vec{f}, \ \vec{\text{curl}} \ \phi), \end{array} \tag{16}$$

where

$$a(\psi, \psi, \phi) = \int_\Omega A(\psi)(\vec{\triangle}\psi \cdot \vec{\triangle}\phi) d\Omega, \tag{17}$$

$$b(\psi, \xi, \phi) = \int_\Omega \triangle\psi(\xi_y \phi_x - \xi_x \phi_y) d\Omega, \tag{18}$$

$$(\vec{f},\ \vec{\text{curl}}\ \phi) = \int_\Omega (f_1 \phi_y - f_2 \phi_x) d\Omega. \tag{19}$$

The above weak formulation is analogous to the weak form of the streamfunction equation of the Navier-Stokes equations (13)

$$\begin{aligned} &\text{Find} \quad \psi \in V \quad \text{such that for all} \quad \phi \in V \\ &\varepsilon_0 \tilde{a}(\psi, \phi) + b(\psi, \psi, \phi) = (\vec{f},\ \vec{\text{curl}}\ \phi), \end{aligned} \tag{20}$$

where

$$\tilde{a}(\psi, \phi) = \int_\Omega (\vec{\triangle}\psi \cdot \vec{\triangle}\phi) d\Omega. \tag{21}$$

The standard weak form of (13) is given by

$$\begin{aligned} &\text{Find} \quad \psi \in V \quad \text{such that for all} \quad \phi \in V \\ &\varepsilon_0 \tilde{\tilde{a}}(\psi, \phi) + b(\psi, \psi, \phi) = (\vec{f},\ \vec{\text{curl}}\ \phi), \end{aligned} \tag{22}$$

where

$$\tilde{\tilde{a}}(\psi, \phi) = \int_\Omega \triangle\psi \triangle \phi d\Omega. \tag{23}$$

It makes no difference whether one uses (20) or (22) because $\tilde{a}(\psi, \phi) = \tilde{\tilde{a}}(\psi, \phi)$ for all $\psi, \phi \in V$. Note that

$$b(\psi, \phi, \phi) = 0 \quad \text{for all} \quad \psi, \phi \in W_0^{2,q}(\Omega), \tag{24}$$

and

$$b(\psi, \phi, \xi) = -b(\psi, \xi, \phi) \quad \text{for all} \quad \psi, \phi, \xi \in W_0^{2,q}(\Omega), \tag{25}$$

and

$$a(\psi, \phi, \xi) = \varepsilon_0 \tilde{a}(\phi, \xi) + \varepsilon_1 \overline{a}(\psi, \phi, \xi),$$

where

$$\overline{a}(\psi, \phi, \xi) = \int_\Omega \parallel \vec{\triangle}\psi \parallel^{q-2} \vec{\triangle}\phi \cdot \vec{\triangle}\xi \ d\Omega. \tag{26}$$

4 Equivalence Forms

Our aim in this section is to prove that the two weak forms (14) and (16) are equivalent. In [25] and [24], existence of the weak solution for problem (14) has been shown. Many uniqueness results for problem (14) can be found in [10] and [9]. Owing to this equivalence, all existence and uniqueness results for the problem (14) carry over to problem (16).

Let us first express the nonlinear term $b(\vec{u},\vec{u},\vec{v})$ in terms of streamfunction, observe

$$\hat{b}(\vec{u},\vec{u},\vec{v}) = \int_\Omega \text{curl } \vec{u}(u_1v_2 - u_2v_1)dx \quad \forall \quad \vec{u} = (u_1,u_2)^T, \vec{v} = (v_1,v_2)^T \in \hat{V}_q.$$

This can be obtained from the following equations

$$\begin{aligned} u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} &= \frac{1}{2}\frac{\partial}{\partial x}(u_1^2+u_2^2) - u_2 \text{ curl } \vec{u}, \\ u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} &= \frac{1}{2}\frac{\partial}{\partial x}(u_1^2+u_2^2) - u_1 \text{ curl } \vec{u}. \end{aligned}$$

Now, integration by parts and eliminating $\vec{\text{grad}}(\| \vec{u} \|^2)$ give

$$a(\psi,\psi,\phi) = b(\vec{u},\vec{u},\vec{v}) \quad \forall \quad \vec{u} = \vec{\text{curl}}\ \psi, \quad \vec{v} = \vec{\text{curl}}\ \phi \quad \text{with} \quad \psi,\phi \in V_q. \quad (27)$$

Now, let us express the term $(\hat{A}(\vec{u})\bigtriangledown\vec{u}, \bigtriangledown\vec{v})$ in terms of streamfunction. Note that

$$(\hat{A}(\vec{u})\bigtriangledown\vec{u}, \bigtriangledown\vec{v}) = \int_\Omega \hat{A}(\vec{u})(\bigtriangledown u_1\cdot\bigtriangledown v_1 + \bigtriangledown u_2\cdot\bigtriangledown v_2)dx \quad \forall \quad \vec{u},\vec{v}\in\hat{V}_q, (28)$$

and

$$\hat{A}(\vec{u}) = A(\psi) \quad \forall\vec{u} = \ \vec{\text{curl}}\ \psi, \quad \psi \in V_q. \quad (29)$$

Also, we have

$$\begin{cases} \bigtriangledown u_1\cdot\bigtriangledown v_1 &= \psi_{xy}\phi_{xy} + \psi_{yy}\phi_{yy} \\ \bigtriangledown u_2\cdot\bigtriangledown v_2 &= \psi_{xy}\phi_{xy} + \psi_{xx}\phi_{yy} \end{cases} \quad \forall \quad \vec{u} = \text{curl } \phi, \quad \vec{v} = \text{curl } \psi. \quad (30)$$

Equations (28), (29) and (30) give:

$$a(\psi,\psi,\phi) = (\hat{A}(\vec{u})\bigtriangledown\vec{u}, \bigtriangledown\vec{v}). \quad (31)$$

By definition, we have

$$(\vec{f},\ \vec{\text{curl}}\ \phi) = (\vec{f},\vec{v}) \quad \forall \quad \vec{f}\in\hat{V}', \quad \vec{v} = \vec{\text{curl}}\ \phi, \quad \phi\in V. \quad (32)$$

Thus the forms (14) and (16) are equivalent and we can state the following theorem.

Theorem 1 *Problems (14) and (16) are equivalent in the sense that if $\vec{u}$ is a solution of (14), then the streamfunction ψ of $\vec{u}$ satisfies (16); conversely, if ψ is a solution of (16), then $\vec{u} = \ \vec{curl}\ \psi$ satisfies (14).*

5 Uniqueness

We define some constants and notations as

$$\begin{aligned}
C_f &:= \sup_{\phi\in V(\Omega)} \frac{|\,(f,\ \mathrm{curl}\ \phi)\,|}{\|\,\phi\,\|_V},\\
C_{f,q} &:= \sup_{\phi\in V_q} \frac{|\,(f,\ \mathrm{curl}\ \phi)\,|}{\|\,\phi\,\|_{V_q}},\\
\gamma_q &:= \sup_{\phi\in V_q} \frac{\|\,\phi\,\|_V}{\|\,\phi\,\|_{V_q}},\\
N &:= \sup_{\phi,\xi\in V} \frac{|\,b(\phi,\phi,\xi)\,|}{\|\,\phi\,\|_V\cdot\|\,\xi\,\|_V},\\
N_q &:= \sup_{\phi,\xi\in V_q} \frac{|\,b(\phi,\phi,\xi)\,|}{\|\,\phi\,\|_{V_q}\cdot\|\,\xi\,\|_{V_q}},\\
\hat{N} &:= \sup_{u,v\in\hat{V};u,v\neq 0} \frac{|\,b(u,u,v)\,|}{\|\,u\,\|_{1,2}^2\cdot\|\,v\,\|_{1,2}},\\
\hat{C}_f &:= \sup_{v\in V;v\neq 0} \frac{|\,(f,v)\,|}{\|\,v\,\|_{1,2}}.
\end{aligned} \tag{33}$$

By the assumption on f, b and Ω, the above constants are well-defined. Moreover, we can state the following

Lemma 2 **(**$N = N_q$**)** .

Proof clearly, $N \geq N_q$; on the other hand, $W_0^{2,q}(\Omega)$ is dense in $H_0^2(\Omega)$. So $\forall\phi,\xi \in H_0^2(\Omega)$, there are sequences of elements $\phi_i, \xi_i \in W_0^{2,q}(\Omega)$ such that $\phi_i \to \phi$ and $\xi_i \to \xi$ in $H_0^2(\Omega)$ as $i \to \infty$. Thus,

$$N_q \geq \lim_{i\to\infty} \frac{|\,b(\phi_i,\phi_i,\xi_i)\,|}{\|\,\phi_i\,\|_V^2\cdot\|\,\xi_i\,\|_V} \geq \frac{|\,b(\phi,\phi,\xi)\,|}{\|\,\phi\,\|_V^2\cdot\|\,\xi\,\|_V}.$$

Hence, $N_q = N$. Thus $N = N_q$ ■

Let ψ be a weak solution for the problem (16). Then we have

Lemma 3 **(**$a(\psi,\psi) = \varepsilon_0 \|\,\psi\,\|_V^2 + \varepsilon_1 \|\,\psi\,\|_{V_q}^q$**)** .

Proof

$$
\begin{aligned}
a(\psi,\psi) &= \int_\Omega A(\psi) \parallel \vec{\triangle}\psi \parallel^2 d\Omega, \\
&= \int_\Omega (\varepsilon_0 + \varepsilon_1 \parallel \vec{\triangle}\psi \parallel^{q-2}) \parallel \vec{\triangle}\psi \parallel^2 d\Omega, \\
&= \varepsilon_0 \int_\Omega \parallel \vec{\triangle}\psi \parallel^2 d\Omega + \varepsilon_1 \int_\Omega \parallel \vec{\triangle}\psi \parallel^q \ d\Omega, \\
&= \varepsilon_0 \parallel \psi \parallel_V^2 + \varepsilon_1 \parallel \psi \parallel_{V_q}^q . \quad \blacksquare
\end{aligned}
$$

Setting $\phi = \psi$ in (16), we obtain

$$
\varepsilon_0 \parallel \psi \parallel_V^2 + \varepsilon_1 \parallel \psi \parallel_{V_q}^q = (\vec{f},\ \vec{\text{curl}}\ \psi). \tag{34}
$$

We then have the following *a priori* estimates.

Theorem 4 *For any weak solution $\psi \in V_q$ of (16), we have*

$$
\parallel \psi \parallel_{V_q}^{q-1} \leq C_{fq}/\varepsilon_1, \tag{35}
$$

and

$$
\parallel \psi \parallel_V \leq R_q(C_f). \tag{36}
$$

Here, R_q is defined as the inverse function of $S_q : (0, +\infty) \to R$

$$
S_q(x) := \varepsilon_0 x + \varepsilon_1 \gamma_q^{-q} \cdot x^{q-1}, \quad \textit{for} \quad x > 0.
$$

Proof Notice that for any $x > 0$

$$
S_q'(x) = \varepsilon_0 + \varepsilon_1 (q-1) \gamma_q^{-q} \cdot x^{q-2}, \quad \text{for} \quad x > 0.
$$

Thus the existence of the function R_q is assumed.

Now (34) implies

$$
\varepsilon_1 \parallel \psi \parallel_{V_q}^q \leq \varepsilon_0 \parallel \psi \parallel_V^q + \varepsilon_1 \parallel \psi \parallel_{V_q}^q = \mid (\vec{f},\ \vec{\text{curl}}\ \psi) \mid,
$$

which gives

$$
\varepsilon_1 \parallel \psi \parallel_{V_q}^{q-1} \leq \frac{\mid (\vec{f},\ \vec{\text{curl}}\ \psi) \mid}{\parallel \psi \parallel_{V_q}} \leq \sup_{\psi \in V_q; \psi \neq 0} \frac{\mid (\vec{f},\ \vec{\text{curl}}\ \psi) \mid}{\parallel \psi \parallel_{V_q}}.
$$

Now, we have

$$
\parallel \psi \parallel_{V_q}^{q-1} \leq \frac{C_{fq}}{\varepsilon_1}.
$$

To prove, (36), we rewrite (34) as

$$
\varepsilon_0 \parallel \psi \parallel_V + \varepsilon_1 \gamma_q^{-q} \parallel \psi \parallel_V^1 \leq C_f.
$$

i.e.

$$S_q(\| \psi \|_V) \leq C_f,$$

which implies (36). ■

Remark 5 *For $q = 3$, an explicit expression of R_3 can be obtained as*

$$R_3(y) = -\frac{1}{2}\varepsilon_1^{-1}\gamma_3^3[\varepsilon_0 - (\varepsilon_0^2 + 4\varepsilon_1\gamma_3^{-3})^{1/2}],$$

or

$$S_3(y) = \frac{2y}{\varepsilon_0 + (\varepsilon_0^2 + 4\varepsilon_1\gamma_3^{-3}y)^{1/2}}.$$

In general, there is no explicit expression for the function S_q.

The following theorem and its proof can be found in [9,10].

Theorem 6 *Assume that the following condition holds*

$$\hat{N}R_q(\hat{C}_f) \leq \varepsilon_0 \quad [or \quad \hat{C}_f \leq S_q(\frac{\varepsilon_0}{\hat{N}})].$$

Then problem (14) has a unique solution.

Theorem (1) and Theorem (6) give the following theorem which states existence and uniqueness of the problem (16).

Theorem 7 *Assume that the following condition holds*

$$NR_q(C_f) \leq \varepsilon_0 \quad [or \quad C_f \leq S_q(\frac{\varepsilon_0}{N})]. \tag{37}$$

Then problem (16) has a unique solution.

Remark 8 *It is known that the streamfunction formulation of the Navier-Stokes equations (20) has a unique solution [see [11,16,17]] whenever*

$$NC_f/\varepsilon_0 < 1. \tag{38}$$

From the definition of the function S_q, we have

$$S_q(y) = \varepsilon_0 y + \varepsilon_1\gamma_q^{-q} \cdot y^{q-1} > \varepsilon_0 y \quad for \quad y > 0, \tag{39}$$

$$S_q(y) > \varepsilon_0 y \quad for \quad y > 0. \tag{40}$$

The monotonicity of S_q and (40) give

$$R_q(y) < y/\varepsilon_0.$$

Equation (37) is less restrictive for $\varepsilon_0 = \frac{1}{Re}$ than (38). In other words, we can find a value of Re which satisfies (37) meaning that we have a guarantee *for existence of a unique solution for the Ladyzhenskaya equations. Whereas, the same value of Re does not satisfy (38) meaning that we do not have a guarantee for existence of a unique solution for the Navier-Stokes equations.*

6 Discretization

In this section, we present a discretized version of (16) and some applicable finite element spaces. We also study the existence and uniqueness of this discretized version. We start by looking at the streamfunction equation of the Navier-Stokes equations which has been studied in [5,6,11–13,26].

We shall give some examples of finite element spaces for the streamfunction formulation. We will impose boundary conditions by setting all the degrees of freedom at the boundary nodes to be zero and the normal derivative equal to zero at all vertices and nodes on the boundary. The inclusion $X^h \subset W_0^{2,q}(\Omega)$ requires the use of finite element functions that are continuously differentiable over Ω. We list below four finite element spaces which could be used for solving the streamfunction equation of the Ladyzhenskaya equations.

- **Argyis triangle:** The functions are quintic polynomials within each triangle and the 21 degrees of freedom are chosen to be the function value and the first and second derivatives at the vertices, and the normal derivative at the midsides.
- **Clough-Tocher:** Here we subdivide each triangle into three triangles by joining the vertices to the centroid. In each of the smaller triangles, the functions are cubic polynomials. There are then 30 degrees of freedom needed to determine the three different cubic polynomials associated with the three triangles. Eighteen of these are used to ensure that, within the big triangle. the functions are continuously differentiable. The remaining 12 degrees of freedom are chosen to be the function values and the first derivatives at the vertices and the normal derivative at the midsides.
- **Bogner-Fox-Schmidt rectangle:** The functions are bicubic polynomials within each rectangle. The degrees of freedom are chosen to be the function value, the first derivatives, and the mixed second derivative at the vertices. We set the function and the normal derivative values equal to zero at all vertices on the boundary.
- **Bicubic Spline rectangle:** The functions are the product of cubic splines. These functions are bicubic polynomials within each rectangle, are twice continuously differentiable over Ω, and their degrees of freedom are the function values at the nodes (plus some additional ones on the boundary).

Let $X^h \subset V_q$ denotes a conforming finite element space. We approximate (16) by the following discrete problem

$$\begin{aligned}&\text{Find} \quad \psi^h \in X^h \quad \text{such that for all} \quad \phi^h \in X^h,\\ &a(\psi^h, \psi^h, \phi^h) + b(\psi^h, \psi^h, \phi^h) = (\vec{f}, \ \vec{\text{curl}}\ \phi^h).\end{aligned} \tag{41}$$

Then, we introduce the following new constants. We define

$$N_h := \sup_{\psi,\phi,\xi \in X^h; \psi,\phi,\xi \neq 0} \frac{b(\psi, \psi, \phi)}{\| \psi \|_V \cdot \| \phi \|_V \cdot \| \xi \|_V}, \tag{42}$$

$$C_{fh} := \sup_{\phi \in X^h; \phi \neq 0} \frac{(f, \ \text{curl}\ \phi)}{\| \phi \|_V}. \tag{43}$$

Remark 9 *By density arguments, it can be shown that*

$$\lim_{h \to 0} N_h = N, \quad \lim_{h \to 0} C_{fh} = C_f. \tag{44}$$

Let us give the following lemma, which will be used later. Lemma (10) can be proofed using the method of [7]. Now we will show that the problem (41) is well posed.

Lemma 10 *For a given number $q > 2$, let $\overline{a}(\cdot, \cdot, \cdot)$ be as defined in (26). Then,*

$$\exists\ \alpha > 0,\ \forall\ \psi, \phi \in W_0^{2,q}(\Omega),\ \overline{a}(\psi, \psi, \psi - \phi) - \overline{a}(\phi, \phi, \psi - \phi) \geq \alpha \| \psi - \phi \|_{V_q}^q, \tag{45}$$

$$\begin{aligned}&\exists\ \ M > 0, \quad \forall\ \psi, \phi \in W_0^{2,q}(\Omega),\\ &| \overline{a}(\psi, \psi, \xi) - \overline{a}(\phi, \phi, \xi) | \leq M \| \psi - \phi \|_{V_q} \Big(\| \psi \|_{V_q} + \| \phi \|_{V_q}\Big)^{q-2} \| \xi \|_{V_q},\end{aligned} \tag{46}$$

Proof Let us introduce the auxiliary function

$$\begin{aligned}\gamma_1 : (\hat{\xi}, \hat{\eta}) \in \mathcal{O} &= \Big\{(\hat{\xi}, \hat{\eta}) \in R^4 \times R^4 :\ \hat{\xi} \neq \hat{\eta}\Big\}\\ \gamma_1 : (\hat{\xi}, \hat{\eta}) &= \frac{\Big(\| \hat{\xi} \|^{q-2}\ \hat{\xi} - \| \hat{\eta} \|^{q-2}\ \hat{\eta}\Big) \cdot \Big(\hat{\xi} - \hat{\eta}\Big)}{\| \hat{\xi} - \hat{\eta} \|^q},\end{aligned} \tag{47}$$

where $\cdot$ denotes the Euclidean inner-product in the space R^4, it has been shown in [[7], pages 319-320] for the case R^2 that

$$\exists\ \alpha > 0, \quad \forall\ (\hat{\xi}, \hat{\eta}) \in \mathcal{O}, \quad \gamma_1(\hat{\xi}, \hat{\eta}) \geq \alpha. \tag{48}$$

Equation (48) remains valid in the case of R^4. However, its proof requires more technicalities.
Setting $\hat{\xi} = \vec{\triangle}\phi$ in (48) and using the definition of the function γ_1 in (47), imply (45).

To prove the second relation (46), we introduce the auxiliary function

$$
\begin{aligned}
\gamma_2 : (\hat{\xi}, \hat{\eta}) \in \mathcal{O} &= \Big\{(\hat{\xi}, \hat{\eta}) \in R^4 \times R^4 : \ \hat{\xi} \neq \hat{\eta}\Big\} \\
\gamma_2 : (\hat{\xi}, \hat{\eta}) &= \frac{\Big(\| \hat{\eta} \|^{q-2} \hat{\eta} - \| \hat{\xi} \|^{q-2} \hat{\xi}\Big) \cdot \Big(\hat{\xi} - \hat{\eta}\Big)}{\| \hat{\eta} - \hat{\xi} \Big(\| \hat{\eta} \| + \| \hat{\xi} \|\Big)^{q-2}},
\end{aligned}
\tag{49}
$$

and it has been shown in [[7], pages 320-321] for the case of R^2 that

$$
\exists\ M > 0, \quad \forall\ (\hat{\xi}, \hat{\eta}) \in \mathcal{O}, \quad \gamma_2(\hat{\xi}, \hat{\eta}) \leq M. \tag{50}
$$

Equation (50) remains valid in the case of R^4. However, its proof requires more technicalities.
As a consequence of (50), we have

$$
\forall \hat{\xi}, \hat{\eta} \in R^4, \quad \| \hat{\eta} \|^{q-2} \hat{\eta} - \| \hat{\xi} \|^{q-2} \hat{\xi} \| \leq M \| \hat{\eta} - \hat{\xi} \| \Big(\| \hat{\eta} \| + \| \hat{\xi} \|\Big)^{q-2}. \tag{51}
$$

Now, the left-hand side of the inequality (46) can be written as

$$
\begin{aligned}
| \overline{a}(\psi, \psi, \xi) - \overline{a}(\phi, \phi, \xi) | &= | \int_\Omega \Big(\| \vec{\triangle}\psi \|^{q-2} \vec{\triangle}\psi - \| \vec{\triangle}\phi \|^{q-2} \vec{\triangle}\phi\Big) \cdot \vec{\triangle}\xi d\Omega \,|, \\
&\leq \int_\Omega \| \vec{\triangle}\psi \|^{q-2} \vec{\triangle}\psi - \| \vec{\triangle}\phi \|^{q-2} \vec{\triangle}\phi \| \vec{\triangle}\xi \| \, d\Omega.
\end{aligned}
$$

Using the inequality (51) gives

$$
\begin{aligned}
&\leq M \int_\Omega \| \vec{\triangle} - \vec{\triangle}\phi \| \Big(\| \vec{\triangle}\psi \| + \| \vec{\triangle}\phi \|\Big)^{q-2} \| \vec{\triangle}\xi \| \, d\Omega, \\
&\leq M \| \psi - \phi \|_{V_q} \Big\{\int_\Omega \Big(\| \vec{\triangle}\psi \| + \| \vec{\triangle}\phi \|\Big)^q d\Omega\Big\}^{\frac{q-2}{q}} \| \xi \|_{V_q}, \\
&\leq M \| \psi - \phi \|_{V_q} \Big(\| \psi \|_{V_q} + \| \phi \|_{V_q}\Big)^{q-2} \| \xi \|_{V_q}. \quad \blacksquare
\end{aligned}
$$

Theorem 11 *The solution to (41) exists and satisfies*

$$
\| \psi^h \|_V \leq R_q(C_{fh}), \tag{52}
$$

$$
\| \psi^h \|_{V_q} \leq (C_{fh}\gamma_q\varepsilon_1^{-1})^{\frac{1}{q-1}}. \tag{53}
$$

Suppose

$$
N_h R_q(C_{fh}) \leq \varepsilon_0 \quad [or \ \ C_{fh} \leq S_q(\varepsilon_0/N_h)]. \tag{54}
$$

Then, the solution ψ^h to (41) is unique.

Proof For all $\phi^h \in X^h$, let us define a mapping $\mathcal{F} : X^h \to X^h$ satisfying

$$
(\mathcal{F}(\phi^h), \xi^h) := a(\phi^h, \phi^h, \xi^h) + b(\phi^h, \phi^h, \xi^h) - (\vec{f}, \ \vec{\text{curl}}\ \xi^h).
$$

Taking $\xi^h = \phi^h$, we get

$$\begin{aligned}
(\mathcal{F}(\phi^h), \phi^h) &= a(\phi^h, \phi^h, \phi^h) - (\vec{f},\ \vec{\text{curl}}\ \phi^h),\\
&= \varepsilon_0 \parallel \phi^h \parallel_V^2 + \varepsilon_1 \parallel \phi^h \parallel_{V_q}^q - (\vec{f},\ \vec{\text{curl}}\ \phi^h),\\
&\geq \varepsilon_0 \parallel \phi^h \parallel_V^2 + \varepsilon_1 \gamma_q^{-q} \parallel \phi^h \parallel_V^q - (\vec{f},\ \vec{\text{curl}}\ \phi^h),\\
&\geq \varepsilon_0 \parallel \phi^h \parallel_V^2 + [\varepsilon_1 \gamma_q^{-q} \parallel \phi^h \parallel_V^q - C_{fh} \parallel \phi^h \parallel_V],\\
&= \parallel \phi^h \parallel_V [\varepsilon_0 \parallel \phi^h \parallel_V + \varepsilon_1 \gamma_q^{-q} \parallel \phi^h \parallel_V^{q-1} - C_{fh}],\\
&= \parallel \phi^h \parallel_V [S_q(\parallel \phi^h \parallel_V) - C_{fh}].
\end{aligned}$$

Then

$$(\mathcal{F}(\phi^h), \phi^h) > 0 \quad \text{for} \quad \parallel \phi^h \parallel_V < R_q(C_{fh}).$$

By a fixed-point theorem (see [16]), there exist an element $\psi^h \in X^h$ such that

$$(\mathcal{F}(\phi^h), \phi^h) = 0 \quad \forall \phi^h \in X^h,$$

which means that ψ^h solves (41).

Now, let ψ^h be a solution for the problem (41) and setting $\phi^h = \psi^h$ in (41) give

$$\varepsilon_0 \parallel \psi^h \parallel_V^2 + \varepsilon_1 \parallel \psi^h \parallel_{V_q}^q = (\vec{f},\ \vec{\text{curl}}\ \psi^h). \tag{55}$$

Using (43) and (33) yield

$$\varepsilon_0 \parallel \psi^h \parallel_V^2 + \varepsilon_1 \gamma_q^{-q} \parallel \psi^h \parallel_V^q \leq C_{fh} \cdot \parallel \psi^h \parallel_V,$$

which implies (52).
(55), (43) and (33) imply (53).

Now to show the uniqueness we assume that ψ_1^h and ψ_2^h are two solutions to (41) and set $\xi^h = \psi_1^h - \psi_2^h$. Then, we have for all $\phi^h \in X^h$

$$a(\psi_1^h, \psi_1^h, \phi^h) + b(\psi_1^h, \psi_1^h, \phi^h) = (\vec{f},\ \vec{\text{curl}}\ \phi^h), \tag{56}$$
$$a(\psi_2^h, \psi_2^h, \phi^h) + b(\psi_2^h, \psi_2^h, \phi^h) = (\vec{f},\ \vec{\text{curl}}\ \phi^h). \tag{57}$$

Let $\phi^h = \xi^h = \psi_1^h - \psi_2^h$ and subtract (57) from (56), then

$$\begin{gathered}
a(\psi_1^h, \psi_1^h, \xi^h) - a(\psi_2^h, \psi_2^h, \xi^h) = b(\psi_1^h, \psi_1^h, \xi^h) + b(\psi_2^h, \psi_2^h, \xi^h),\\
\text{LHS} = \text{RHS}.
\end{gathered}$$

By (24), we have

$$\text{RHS} = b(\psi_1^h, \psi_1^h, \psi_2^h) + b(\psi_2^h, \psi_2^h, \psi_1^h).$$

By (25), we have

$$\begin{aligned}\text{RHS} &= b(\psi_1^h, \psi_1^h, \psi_2^h) - b(\psi_2^h, \psi_1^h, \psi_2^h),\\ &= b(\xi^h, \psi_1^h, \psi_2^h).\end{aligned}$$

Since $b(\xi^h, \psi_2^h, \psi_2^h) = 0$, then we have

$$\text{RHS} = b(\xi^h, \xi^h, \psi_2^h).$$

Now, we apply (45) to the LHS to obtain

$$\varepsilon_0 \parallel \xi^h \parallel_V^2 + \alpha\varepsilon_1 \parallel \xi^h \parallel_{V_q}^q \leq b(\xi^h, \xi^h, \psi_2^h).$$

By (33) and (42), we have

$$\varepsilon_0 \parallel \xi^h \parallel_V^2 + \alpha\varepsilon_1 \gamma_q^{-q} \parallel \xi^h \parallel_V^q \leq N_h \parallel \xi^h \parallel_V^2 \parallel \psi_2^h \parallel_V .$$

Using (52) gives

$$\varepsilon_0 \parallel \xi^h \parallel_V^2 + \alpha\varepsilon_1 \gamma_q^{-q} \parallel \xi^h \parallel_V^q \leq N_h R_q(C_{fh}) \parallel \xi^h \parallel_V^2 .$$

After some arrangements, we have

$$\parallel \xi^h \parallel_V^2 \cdot [\varepsilon_0 - N_h R_q(C_{fh}) + \alpha\varepsilon_1 \gamma_q^{-q} \parallel \xi^h \parallel_V^{q-2}] \leq 0,$$

which implies that

$$\parallel \xi^h \parallel_V = 0 \quad \text{or} \quad \parallel \xi^h \parallel_V^{q-2} \leq \frac{\gamma_q^{-q}}{\alpha\varepsilon_1} [N_h R_q(C_{fh}) - \varepsilon_0].$$

Hence, if $N_h R_q(C_{fh}) \leq \varepsilon_0$ [or $C_{fh} \leq S_q(\varepsilon_0/N_h)$] holds, then (41) has a unique solution. ■

Now Theorem (11) and equation (44) give the following theorem

Theorem 12 *Assume that* $NR_q(C_f) < \varepsilon_0$ *holds; then when* h *is small enough, problem (41) has a unique solution.*

Remark 13 *Moreover, by (44), there exists a constant* $\delta > 0$ *such that for* h *small, we have*

$$\varepsilon_0^{-1} N R_q(C_f) \leq 1 - \delta < 1. \tag{58}$$

7 Error Bound

Now that we know that a unique finite element approximation ψ^h is defined by (41), we wish to assess the size of the error $\varepsilon = \psi - \psi^h$, when ψ is the solution of problem (16). In this section, we assume that (54) holds and (58) is valid for h small.

Theorem 14 *Let $X^h \subset V_q$ be a finite element space. Let ψ be the solution to (16) and ψ^h be solution to (41). Then for h sufficiently small, ψ^h satisfies*

$$\| \psi - \psi^h \|_V \leq \hat{C} \inf_{\chi^h \in X^h} \| \psi - \chi^h \|_V, \tag{59}$$

where

$$\hat{C} = \frac{\varepsilon_0 + \varepsilon_1 M \gamma_q^{-2} \{(C_{fq} \varepsilon_1^{-1})^{\frac{1}{q-1}} + (C_{fh} \gamma_q^{-1} \varepsilon_1)^{\frac{1}{q-1}}\}^{q-2} + N\{R_q(C_f) + R_q(C_{fh})\}}{\varepsilon_0 - N R_q(C_f)}.$$

Proof Since $X^h \subset V_q$, (16) holds for all $\chi^h \in X^h$. Now, we have

$$a(\psi, \psi, \chi^h) + b(\psi, \psi, \chi^h) = (\vec{f}, \text{curl } \chi^h) \quad \forall\, \chi^h \in X^h. \tag{60}$$

Subtracting (41) from (60) gives

$$a(\psi, \psi, \chi^h) - a(\psi, \psi, \chi^h) + b(\psi, \psi, \chi^h) - b(\psi, \psi, \chi^h) = 0 \quad \forall\, \chi^h \in X^h. \tag{61}$$

Setting $\chi^h = \psi^h$ in (61) gives

$$\begin{aligned} &a(\psi, \psi, \chi^h - \psi^h) - a(\psi, \psi, \chi^h - \psi^h) + b(\psi, \psi, \chi^h - \psi^h) \\ &- b(\psi, \psi, \chi^h - \psi^h) = 0 \quad \forall\, \chi^h \in X^h. \end{aligned} \tag{62}$$

The first and the second term in (62) can be rewritten as

$$\begin{aligned} &a(\psi, \psi, \chi^h - \psi^h) - a(\psi, \psi, \chi^h - \psi^h) = \\ &\varepsilon_0[\tilde{a}(\psi - \psi^h, \psi - \psi^h) - \tilde{a}(\psi - \psi^h, \psi - \chi^h)] \\ &+ \varepsilon_1[\overline{a}(\psi^h, \psi^h, \psi - \chi^h) - \overline{a}(\psi, \psi, \psi - \chi^h) \\ &+ \overline{a}(\psi, \psi, \psi - \psi^h) - \overline{a}(\psi^h, \psi^h, \psi - \psi^h)]. \end{aligned} \tag{63}$$

The third and the fourth term in (62) can be rewritten as

$$\begin{aligned} &b(\psi, \psi, \chi^h - \psi^h) - b(\psi^h, \psi^h - \psi, \chi^h - \psi^h) = \\ &b(\psi^h - \psi, \psi, \psi^h - \psi) - b(\psi^h, \psi^h - \psi, \chi^h - \psi) - b(\psi^h - \psi, \psi, \chi^h - \psi). \end{aligned} \tag{64}$$

Rearranging terms after using (63) and (64) in (62) gives

$$\begin{aligned} &\varepsilon_0 \tilde{a}(\psi - \psi^h, \psi - \psi^h) + \varepsilon_1[\overline{a}(\psi, \psi, \psi - \chi^h) - \overline{a}(\psi^h, \psi^h, \psi - \chi^h)] \\ &+ b(\psi^h - \psi, \psi, \psi^h - \psi) = \\ &\varepsilon_0 \tilde{a}(\psi - \psi^h, \psi - \chi^h) + \varepsilon_1[\overline{a}(\psi, \psi, \psi - \psi^h) - \overline{a}(\psi^h, \psi^h, \psi - \chi^h)] \\ &+ b(\psi^h, \psi^h - \psi, \chi^h - \psi) + b(\psi^h - \psi, \psi, \chi^h - \psi). \end{aligned} \tag{65}$$

Using (45) in LHS of (65) gives

$$\begin{aligned} \text{LHS of (65)} &\geq \varepsilon_0 \| \psi - \psi^h \|_V^2 - N \| \psi - \psi^h \|_V \| \psi \|_V, \\ &\geq [\varepsilon_0 - N R_q(C_f)] \| \psi - \psi^h \|_V^2 . \end{aligned} \tag{66}$$

Using (46) in RHS of (65) gives

$$\begin{aligned} \text{RHS of (65)} \leq & \varepsilon_0 \parallel \psi - \psi^h \parallel_V \parallel \psi - \chi^h \parallel_V \\ & + \frac{\varepsilon_1 M}{\gamma_q^2} \parallel \psi - \psi^h \parallel_V \left(\parallel \psi \parallel_{V_q} + \parallel \psi^h \parallel_{V_q} \right)^{q-2} \parallel \psi - \chi^h \parallel_V \\ & + N \left(\parallel \psi \parallel_{V_q} + \parallel \psi^h \parallel_{V_q} \right) \parallel \psi - \psi^h \parallel_V \parallel \psi - \chi^h \parallel_V . \end{aligned} \tag{67}$$

Now, (65), (66) and (67) imply

$$\begin{aligned} [\varepsilon_0 - N R_q(C_f)] \parallel \psi - \psi^h \parallel_V \leq & \varepsilon_0 \parallel \psi - \chi^h \parallel_V \\ & + \frac{\varepsilon_1 M}{\gamma_q^2} \left(\parallel \psi \parallel_{V_q} + \parallel \psi^h \parallel_{V_q} \right)^{q-2} \parallel \psi - \chi^h \parallel_V \\ & + N \left(\parallel \psi \parallel_{V_q} + \parallel \psi^h \parallel_{V_q} \right) \parallel \psi - \chi^h \parallel_V . \end{aligned}$$

Using (35), (36), (52), (53) and the condition $\varepsilon_0 < N R_q(C_f)$, we get

$$\parallel \psi - \psi^h \parallel_V \leq \hat{C} \parallel \psi - \chi^h \parallel_V \quad \text{for all} \quad \chi^h \in X^h. \tag{68}$$

The conclusion is immediate from (68). ■

As an example, if the Bogner-Fox-Schmidt Rectangles are used, then there exist a positive constant C such that $\parallel \psi - \psi^h \parallel_V \leq Ch^2$. For each of the elements mentioned in section 6, Table (1) shows the error estimates.

Element	Estimate
Argyis triangle	$\parallel \psi - \psi^h \parallel_V = O(h^4)$
Clough-Tocher triangle	$\parallel \psi - \psi^h \parallel_V = O(h^2)$
Bogner-Fox-Schmidt rectangle	$\parallel \psi - \psi^h \parallel_V = O(h^2)$
Bicubic Spline rectangle	$\parallel \psi - \psi^h \parallel_V = O(h^2)$

Table 1: Accuracy of Finite Elements for the Streamfunction Formulation

8 Computational Experiments

We consider the driven cavity problem in the two-dimensional box $[0,1] \times [0,1]$ when the top surface moves with a constant velocity along its length i.e. $u =$

$v = 0$ in all boundaries except $y = 1$, where $u = 1$. Cavity flows have been a subject of study for some time [4,12,13,35]. These flows have been widely used as test cases for validating incompressible fluid dynamics algorithms. The numerical computational in this example was obtained using an IBM NetVista PC with 1.6 Ghz Intel Pentium IV processor running Windows 98 SE. Bogner-Fox-Schmit elements are used with 9×9 grid points and 11×11 grid. We pick one value for the Reynolds number, $Re = 1$. The second viscosity coefficient ε_1 is also chosen to be relatively small compared to Re. We choose $\varepsilon_1 = e^{-15}, e^{-16}, ..., e^{-45}$. In the computation of this problem, we use the following iterative scheme where we linearize the added nonlinear term and then solve the nonlinear system of equations by using Newton's method. Let $\psi^{(0)} \in X^h$ be given; then we define the sequence $\psi^{(n)} \in X^h$ for $n = 1, 2, 3, ...$, to be the solution of the following nonlinear discrete system:

$$\text{Find } \psi^{(n)} \in X^h \text{ such that , for all } \phi \in X^h, \tag{69}$$

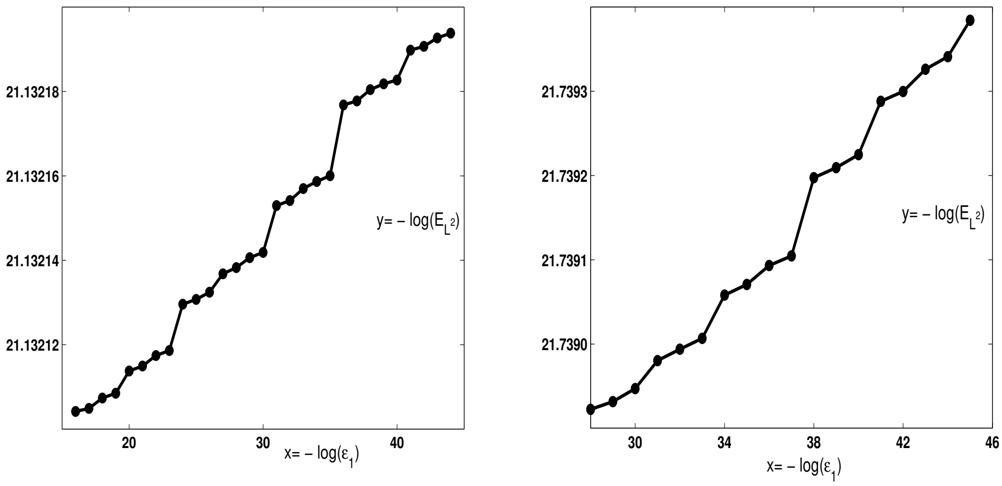
$$a(\psi^{(n)}, \phi) + \hat{a}(\psi^{(n-1)}, \psi^{(n)}, \phi) + b(\psi^{(n)}; \psi^{(n)}, \phi) = (\vec{f}, \vec{curl}\phi). \tag{70}$$

The resulting matrix from each iteration is nonsymmetric whose symmetric part is positive definite. Moreover, the resulting matrix is sparse. The suggested linear solver for such system is any Conjugate Gradient alike method. We choose the Bi Conjugate Gradient Stabilized method (Bi-CGSTAB) ,(see Templates [2]), to solve the linear system resulting from each Newton's iterate. Bi-CGSTAB was developed to solve nonsymmetric systems. The stopping criteria for the problem is

$$\| \psi^{(n+1)} - \psi^{(n)} \| \leq TOL \quad \text{and} \quad \| residual \| \leq TOL,$$

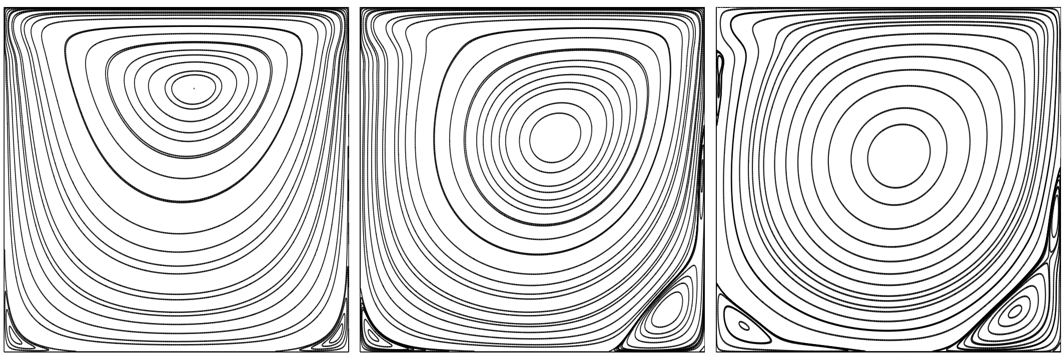
with $TOL = 1.0e-5$ where the above two norms are in the discrete L_2-norm. Our computations show that we get a stable approximation of the unique solution for (12). The Cavity problem is solved using both the streamfunction equation of the Navier-Stokes model (13) and the streamfunction equation of the Ladyzhenskaya model (12). The numerical computations were performed for different choices of the second viscosity parameter ε_1 and different sizes of triangulations. Each time, we evaluate $\| \psi_L^h - \psi_N^h \|$ where ψ_L^h is the approximate solution of the streamfunction equation of the Ladyzhenskaya model and ψ_N^h is the approximate solution of the streamfunction equation of the Navier-Stokes model. Then we interpolate the results from these cases to obtain the graphs in Figure(1a) and Figure(1b). The graphs are produced in the logarithmic coordinate system so that we can see more clearly the fact that the difference in the discrete L_2-norm between solutions does tend to zero as ε_1 tends to zero.

The second computational experiment in this section was obtained using a TOSHIBA Satellite Pro with Intel Mobile CPU 1.7GHz running Windows XP. Bogner-Fox-Schmit elements are used with 17x17 grid points. We choose $\varepsilon_1 = 1e - 20$ and $q = 4$. We compute an approximate solution for Re = 1, 10,



(a) Number of elements = 64 (b) Number of elements = 100

Fig. 1. Differences between the approximate solution of the streamfunction equation of a Ladyzhenskaya and the streamfunction equation of the NSE. Re = 1.0, q=4. E_{L^2} = Difference in the L^2 norm.



(a) Re = 1 (b) Re = 10 (c) Re = 30

Fig. 2. Cavity Problem : Streamlines for different values of Reynolds numbers

30. Figures (2a,2b,2c) display streamfunction contours. We can see that the top right corner, where the moving wall moves towards the stationary wall, shows that the streamfunction contours are very smooth. It is also seen that the number of vortices in the bottom right corner increases as the Reynolds number increases.

9 Conclusion

A weak formulation for the streamfunction equation of the Ladyzhenskaya equations was discussed. The discretized version was also studied and some finite element spaces applicable with it are provided. At some specific values of the parameters, the Ladyzhenskaya equations become the Smagorinsky model [36]. The Smagorinsky model, which is widely recognized, is the most popular model in Large Eddy Simulation (LES). LES has received many scientific development and it is currently viewed as the most accurate and promising approach to the simulation of turbulence. Then there has been a very active search for better LES models. Recent works on LES are accomplished by Layton and his group [15,20,21,28]. Further studies using LES, e.g. LES approach of Hughes called Variational Multiscale Methods (VMM) [19,22,29–31], are very valuable. Further studies including combinary LES approaches and streamfunction form, and investigations of solutions, are subject of a forthcoming study.

In this paper, we consider the two-dimensional streamfunction form of the Ladyzhenskaya equations. One may think of considering the three-dimensional version. Before we discuss this consideration, let us look at the three-dimensional streamfunction form of the Navier-Stokes equations. First, a vector potential Ψ is to be defined. Next, we let $u = \text{curl } \Psi$. Then, we eliminate the vorticity $\omega = \text{curl } u$ from the three-dimensional streamfunction-vorticity equations. Thus we obtain a single vector valued equation for the vector potential Ψ. Following this way, the incompressibility condition div $u = 0$ is also automatically satisfied. Moreover, the pressure is not present. So the two features of the streamfunction form mentioned above are carried over in the three-dimensional version. On the other hand, for the third feature of using the streamfunction form, one unknown to solve for, we can notice that the number of unknowns to solve for is three in the three-dimensional version while there are only four unknowns in velocity-pressure form. In addition, some researchers have attacked three-dimensional vector potential formulation of the Navier-Stokes equations. Further discussion and details can be found in [3,32,34,37]. All of the above issues that occur when using three-dimensional streamfunction form of the Navier-Stokes equations will also occur when considering three-dimensional streamfunction form of the Ladyzhenskaya equations.

Acknowledgements

This research work was undertaken while the author was on sabbatical leave from King Fahd University of Petroleum and Minerals. The author wishes to express his gratitude to the University for providing the financial support while on research attachment at Univesiti Tenaga Nasional, Malaysia. The author would like to express thanks to Prof. William J. Layton, Prof. Qiang Du and Dr. Songul Kaya for their valuable comments on this work.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, New York, 1975.

[2] R. Barrett M. Berry T. Chan J. Demmel J. Donator J. Doncarra V. Eijkhout R. Pozo C. Romine and H. Van dev Vorst. *templates: for the solution of linear systems: building blocks for iterative methods.* e-mail : templates@cs.utk.edu.

[3] A. Bendali J. Dominguez and S. Gallic. A variational approach for the vector potential formulation of the Stokes and Navier-Stokes equations in three dimensional domain. *J.Math.Anal.Appl.*, 107:537–560, 1985.

[4] P. Betts and V. Haroutunian. A stream function finite element solution for two-dimensional natural convection. *SIAM J.sci.Comput.*, 16(4):773–797, 1982.

[5] M. Cayco and R. A. Nicolaides. Finite element technique for optimal pressure recovery from stream function formulation of viscous flows. *Math. Comp.*, 46:371–377, 1986.

[6] M. Cayco and R. A. Nicolaides. Analysis of nonconforming stream function and pressure finite element spaces for the Navier-Stokes equations. *Comp. and Math. Appl*, 8:745–760, 1989.

[7] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* Studies in Mathematics and its Applications. Vol. 4. Amsterdam - New York - Oxford: North-Holland Publishing Company. XIX, 580 p. Dfl. 130.00; $ 56.00 , 1978.

[8] Q. Du, M. D. Gunzburger, and L.Steven Hou. Analysis and finite element approximation of optimal control problems for a Ladyzhenskaya model for stationary, incompressible, viscous flows. *J. Comput. Appl. Math.*, 61(3):323–343, 1995.

[9] Q. Du and M. D. Gunzburger. Finite-element approximations of a Ladyzhenskaya model for stationary incompressible viscous flow. *SIAM J. Numer. Anal.*, 27(1):1–19, 1990.

[10] Q. Du and M. D. Gunzburger. Analysis of a Ladyzhenskaya model for incompressible viscous flow. *J. Math. Anal. Appl.*, 155(1):21–45, 1991.

[11] F. Fairag. A two-level finite element discretization for the stream function form of the Navier-Stokes equations. *Computer Math. Applic.*, 2:117–127, 1998.

[12] F. Fairag. Two level finite element technique for pressure recovery from stream function formulation of the Navier-Stokes equations. In *Multiscale and Multiresolution Methods Theory and Applications,Lecture Notes in Computational Science and Engineering*, pages 297–305. Springer, New York, 2002.

[13] F. Fairag. Numerical computations of viscous, incompressible flow problems using a two-level finite element method. *SIAM J. Sci. Comput.*, 24(6):1919–1929, 2003.

[14] F. Fairag. Finite element computation of a pure-streamfunction equation of a Ladyzhenskaya model of an incompressible fluid. *submitted*, 2005.

[15] G. P. Galdi and W. J. Layton. Approximation of the larger eddies in fluid motions. II: A model for space-filtered flow. *Math. Models Methods Appl. Sci.*, 10(3):343–350, 2000.

[16] V. Girault and P. A. Raviart. *Finite Element Approximation of the Navier-Stokes Equations*, volume 749. Springer, Berlin, 1979.

[17] V. Girault and P. A. Raviart. *Finite Element Methods for the Navier-Stokes Equations : Theory and Algorithms.* Springer, Berlin, 1986.

[18] M. Gunzburger. *Finite Element Method for Viscous Incompressible Flow : A Guide to Theory Practice and Algorithms.* Academic Press, Boston, 1989.

[19] T. J. R. Hughes. The multiscale phenomena: Green's functions, the Dirichlet to Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods. *Computer Methods in Applied Mechanics and Engineering*, 127:387–401, 1995.

[20] T. Iliescu and W.J. Layton. Approximating the larger eddies in fluid motion III: The Boussinesq model for turbulent fluctuations. *An. Stiint. Univ. Al. I. Cuza Iasi, Ser. Noua, Mat.*, 44(2):245–261, 1998.

[21] V. John and W.J. Layton. Analysis of numerical errors in large eddy simulation. *SIAM J. Numer. Anal.*, 40(3):995–1020, 2002.

[22] S. Kaya. *Numerical Analysis of a Variational Multiscale Method for Turbulence.* PhD thesis, University of Pittsburgh, Pittsburgh, PA., 2004.

[23] O. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York, 1969.

[24] O. Ladyzhenskaya. Modification of the Navier-Stokes equations for large velocity gradients. *Boundary value problems of mathematical physics and related aspects of function theory II*, pages 57–69, 1970.

[25] O. Ladyzhenskaya. New equation for the description of the viscous incompressible fluids and solvability in the large of the boundary value problems for them. *Boundary value problems of mathematical physics V*, pages 95–118, 1970.

[26] W. Layton and X. Ye. Two-level discretizations of the stream function form of the Navier-Stokes equations. *Numer. Funct. Anal. Optimization*, 20(9-10):909–916, 1999.

[27] W. Layton. A nonlinear subgridscale model for incompressible viscous flow problems. *SIAM. J. Sci. Comput.*, 17:347–357, 1996.

[28] W. Layton G. Matthies T. Iliescu, V. John and L. Tobiska. A numerical study of a class of LES models. *International Journal of Computational Fluid Dynamics*, 17(1):75–85, 2003.

[29] L. Mazzei T. J. R. Hughes and K. E. Jansen. Large eddy simulation and the variational multiscale method. *Comput. Visual Sci.*, 3:47–59, 2000.

[30] A. A. Oberai T. J. R. Hughes, L. Mazzei and A.Wray. The multiscale formulation of large eddy simulation: decay of homogenous isotropic turbulence. *Physics of Fluids*, 13:505–512, 2001.

[31] A. A. Oberai T. J. R. Hughes and L. Mazzei. Large eddy simulation of turbulent channel flows by the variational multiscale method. *Physics of Fluids*, 13:1784–1799, 2001.

[32] V. Ruas. Some nonstandard finite element methods for the numerical solution of viscous flow problems. *Tenth International Conference on Numerical Methods in Fluid Dynamics (Ed. by F. Zhuang and Y. Zhu)*, Springer, Berlin, 538–544, 1986.

[33] F. Schieweck. On the order of two nonconforming finite element approximations of upwind type for the Navier-Stokes equations. Proceeding of the international Workshop on Numerical Methods for the Navier-Stokes Equations, Heidelberg, October 25-28, 1993, series Notes on Numerical Fluid Mechanics, Vieweg-Verlag.

[34] A. Sherif and M. Hafez. Computation of three-dimensional transonic flows using two stream functions. *AIAA Paper, AIAA, New York*, 83-1948, 1983.

[35] C.T. Shin U. Ghia, K.N. Ghia. High Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method. *J. Compt. Phys.*, 48:387–411, 1982.

[36] J. Smagorinsky. General circulation experiments with the primitive equations,Part I: the basic experiment. *Month. Weath. Rev.*, 91(3):99–164, 1963.

[37] R. Verfurth. Mixed finite element approximation of the vector potential. *Numer. Math.*, 50:685–695, 1987.