

King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 202 Final Exam
Semester II, 2008- (072)
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Name:	KEY
ID:	KEY

Q		Points
1		20
2		20
3		32
4		32
5		24
6		10
7		10
8		15
9		10
10		10
11		15
12		10
13		10
14		10
15		22
Bonus		10
Total		250

(Q1) Find the general solution of the system: $X' = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} X$

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = (\lambda-2)^2$$

$\lambda = 2, 2$ are eigenvalues for $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Eigenvector:

$$[A - 2I | 0] = \begin{bmatrix} 0 & 0 & | & 0 \\ 1 & 0 & | & 0 \end{bmatrix} \Rightarrow k_1 = 0 \Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 \end{bmatrix}$$

Hence, $K = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector

$$X_1 = K e^{\lambda t} \Rightarrow X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t}$$

Find P:

$$[A - 2I | K] = \begin{bmatrix} 0 & 0 & | & 0 \\ 1 & 0 & | & 1 \end{bmatrix} \Rightarrow p_1 = 1 \Rightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ p_2 \end{bmatrix}$$

choose $P = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$X_2 = K t e^{\lambda t} + P e^{\lambda t} \Rightarrow X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

$$X = C_1 X_1 + C_2 X_2$$

(Q2) Find the general solution of the system:

$$X' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} X$$

$$P(\lambda) = |A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + (1) \begin{vmatrix} 0 & 1 \\ 1-\lambda & 0 \end{vmatrix}$$

$$= (-\lambda)(\lambda^2 - \lambda) + (\lambda - 1) = -\lambda^2(\lambda - 1) + (\lambda - 1) = -(\lambda - 1)(\lambda^2 - 1)$$

$$= -(\lambda - 1)^2(\lambda + 1) \Rightarrow \lambda = 1, 1, -1 \text{ eigenvalues} \quad \triangle 5$$

$$\underline{\lambda = -1}: [A + I | 0] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right] \xrightarrow[\frac{1}{2}R_2]{-R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow k_1 = -k_3, k_2 = 0 \Rightarrow \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -k_3 \\ 0 \\ k_3 \end{bmatrix} \Rightarrow K = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \triangle 5$$

$$\underline{\lambda = 1}: [A - I | 0] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right] \xrightarrow[-R_1]{-R_1 + R_2} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \Rightarrow k_1 = k_3$$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow K_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, K_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ are eigenvectors } \lambda = 1$$

$$X = \begin{bmatrix} X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t \end{bmatrix} \quad \triangle 5$$

$$X = C_1 X_1 + C_2 X_2 + C_3 X_3$$

(Q3) Use Variation of Parameters to solve the system:

$$X' = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} \sin t \\ 1 \end{pmatrix}$$

Hint: $\begin{bmatrix} 1+i \\ 1 \end{bmatrix}$ is an eigenvector for the matrix A associated with the eigenvalue $\lambda = i$

$$\lambda = i \Rightarrow \alpha = 0, \beta = 1$$

$$K = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} i \Rightarrow \beta_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \beta_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X_1 = (\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sin t) = \begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix} \quad \triangle$$

$$X_2 = (\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin t) = \begin{bmatrix} \cos t + \sin t \\ \sin t \end{bmatrix} \quad \triangle$$

$$\text{Now, } \Phi(t) = \begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{bmatrix}$$

$$|\Phi(t)| = \cos t \sin t - \sin^2 t - \cos^2 t - \cos t \sin t = -1 \quad \triangle$$

$$\Phi^{-1}(t) = \frac{1}{|\Phi(t)|} \begin{bmatrix} \sin t & -\cos t - \sin t \\ -\cos t & \cos t - \sin t \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t + \sin t \\ \cos t & \sin t - \cos t \end{bmatrix} \quad \triangle$$

$$\Phi^{-1}(t) \cdot F(t) = \begin{bmatrix} \sin t + \cos t \\ \sin t - \cos t \end{bmatrix} \Rightarrow U = \int \Phi^{-1} F dt = \begin{bmatrix} -\cos t + \sin t \\ -\cos t - \sin t \end{bmatrix} \quad \triangle$$

$$\text{Now, } X_p = \Phi U = \begin{bmatrix} \cos t - \sin t & \cos t + \sin t \\ \cos t & \sin t \end{bmatrix} \begin{bmatrix} -\cos t + \sin t \\ -\cos t - \sin t \end{bmatrix} \quad \triangle$$

$$= \begin{bmatrix} -(\sin t - \cos t)^2 - (\sin t + \cos t)^2 \\ -\cos^2 t + \sin t \cos t - \sin^2 t - \sin t \cos t \end{bmatrix} \quad \triangle$$

$$X = \begin{bmatrix} -(1 - 2\sin t \cos t) - (1 + 2\sin t \cos t) \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

The general solution:

$$X(t) = c_1 \begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + \sin t \\ \sin t \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

\triangle

(Q4) $x=0$ is a regular singular point of the DE

$$2xy'' - y' + 2y = 0$$

$$2xy'' - 2y = 0$$

Use the method of Frobenius to obtain two linearly independent series solutions about $x=0$.

standard form: $y'' - \frac{2}{x}y = 0 \Rightarrow P=0, Q=-\frac{2}{x}$

$a_0=0, b_0=0$

indicial eq: $r(r-1)=0$

indicial roots: $r_1=1, r_2=0$

Case: $r_2=0$, $y = \sum_{n=0}^{\infty} c_n x^n$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$$

$$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots$$

$$xy'' = 2c_2 x + 6c_3 x^2 + 12c_4 x^3 + 20c_5 x^4 + 30c_6 x^5 + \dots$$

$$-2y = -2c_0 - 2c_1 x - 2c_2 x^2 - 2c_3 x^3 - 2c_4 x^4 - 2c_5 x^5 - \dots$$

$$\text{LHS} = -2c_0 + (2c_2 - 2c_1)x + (6c_3 - 2c_2)x^2 + (12c_4 - 2c_3)x^3 + (20c_5 - 2c_4)x^4 + (30c_6 - 2c_5)x^5 + \dots = 0$$

$$c_0 = 0, c_2 = c_1, c_3 = \frac{1}{3}c_2 = \frac{1}{3}c_1, c_4 = \frac{1}{6}c_3 = \frac{1}{18}c_1$$

$$c_5 = \frac{1}{10}c_4 = \frac{1}{180}c_1, c_6 = \frac{1}{15}c_5 = \frac{1}{2700}c_1$$

$$y_1 = c_1 x + c_1 x^2 + \frac{1}{3}c_1 x^3 + \frac{1}{18}c_1 x^4 + \frac{1}{180}c_1 x^5 + \frac{1}{2700}c_1 x^6 + \dots$$

$$y_1(x) = x + x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 + \frac{1}{180}x^5 + \frac{1}{2700}x^6 + \dots$$

We need to use section 4.2 to find the second linearly independent solution

$$y_2(x) = y_1(x) \int \frac{K(x)}{[y_1(x)]^2} dx$$

$K(x) = e^{-\int p dx} = e^{-\int 0 dx} = 1$

$y_2 =$

$$y_1^2 = (x + x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 + \frac{1}{180}x^5 + \dots)(x + x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 + \frac{1}{180}x^5 + \dots)$$

$$= x^2 + 2x^3 + (\frac{1}{3} + 1 + \frac{1}{3})x^4 + (\frac{1}{18} + \frac{1}{3} + \frac{1}{3} + \frac{1}{18})x^5 + (\frac{1}{180} + \frac{1}{18} + \frac{1}{9} + \frac{1}{18} + \frac{1}{180})x^6 + \dots$$

$$\boxed{5} = x^2 + 2x^3 + \frac{5}{3}x^4 + \frac{7}{9}x^5 + \frac{7}{30}x^6 + \dots$$

$$x^2 + 2x^3 + \frac{5}{3}x^4 + \frac{7}{9}x^5 + \frac{7}{30}x^6 + \dots$$

$$x^{-2} - 2x^{-1} + \frac{7}{3} - \frac{19}{9}x + \frac{149}{90}x^2$$

$$1 + 2x + \frac{5}{3}x^2 + \frac{7}{9}x^3 + \frac{7}{30}x^4 + \dots$$

$$\frac{-2x - \frac{5}{3}x^2 - \frac{7}{9}x^3 - \frac{7}{30}x^4 + \dots}{-2x - 4x^2 - \frac{10}{3}x^3 - \frac{14}{9}x^4 + \dots}$$

$$\frac{\frac{7}{3}x^2 + \frac{23}{9}x^3 + \frac{119}{90}x^4 + \dots}{\frac{7}{3}x^2 + \frac{14}{3}x^3 + \frac{35}{9}x^4 + \dots}$$

$$\frac{-\frac{19}{9}x^3 - \frac{231}{90}x^4 + \dots}{-\frac{19}{9}x^3 - \frac{38}{9}x^4 + \dots}$$

$$\frac{\frac{149}{90}x^4}{\frac{149}{90}x^4}$$

$$\boxed{\frac{1}{y_1^2} = x^{-2} - 2x^{-1} + \frac{7}{3} - \frac{19}{9}x + \frac{149}{90}x^2 + \dots}$$

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$$\int \frac{k}{y_1^2} dx = \int (x^{-2} - 2x^{-1} + \frac{7}{3} - \frac{19}{9}x + \frac{149}{90}x^2 + \dots) dx$$

$$= -x^{-1} - 2 \ln x + \frac{7}{3}x - \frac{19}{18}x^2 + \frac{149}{270}x^3 + \dots$$

$$\boxed{\int \frac{k}{y_1^2} dx = -x^{-1} - 2 \ln x + \frac{7}{3}x - \frac{19}{18}x^2 + \frac{149}{270}x^3 + \dots}$$

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$$y_2 = y_1 \int \frac{k}{y_1^2} dx = (x + x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 + \frac{1}{180}x^5 + \dots)(-x^{-1} - 2 \ln x + \frac{7}{3}x - \frac{19}{18}x^2 + \frac{149}{270}x^3 + \dots)$$

$$= -2y_1(x) \ln x + (x + x^2 + \frac{1}{3}x^3 + \frac{1}{18}x^4 + \frac{1}{180}x^5 + \dots)(-x^{-1} + \frac{7}{3}x - \frac{19}{18}x^2 + \frac{149}{270}x^3 + \dots)$$

$$= -2y_1 \ln x + \left[-1 - x + (\frac{7}{3} + \frac{1}{3})x^2 + (-\frac{19}{18} + \frac{7}{3} - \frac{1}{18})x^3 + (\frac{149}{270} - \frac{19}{18} + \frac{7}{9} - \frac{1}{180})x^4 + \dots \right]$$

$$= -2y_1 \ln x + \left[-1 - x + 2x^2 + \frac{11}{9}x^3 + \frac{29}{108}x^4 + \dots \right]$$

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(Q5) Find two power series solutions of the given DE about the ordinary point $x = 0$

$$y'' + (\frac{\sin x}{\cos x})y = 0 \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$x = 0$ is an ordinary point, so the solution $y = \sum c_n x^n$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$$

$$y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots$$

$$(\sin x)y = (x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots)$$

$$= c_0 x + (c_1)x^2 + (c_2 - \frac{1}{6}c_0)x^3 + (c_3 - \frac{1}{6}c_1)x^4 + (c_4 - \frac{1}{6}c_2 + \frac{1}{120}c_0)x^5 + \dots$$

$$\text{LHS} = 2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + (20c_5 + c_2 - \frac{1}{6}c_0)x^3$$

$$\triangle + (30c_6 + c_3 - \frac{1}{6}c_1)x^4 + \dots = 0$$

$$\Rightarrow c_2 = 0, c_3 = -\frac{1}{6}c_0, c_4 = -\frac{1}{12}c_1, c_5 = \frac{1}{120}c_0 - \frac{1}{20}c_2 = \frac{1}{120}c_0$$

$$c_6 = -\frac{1}{30}c_3 + \frac{1}{180}c_1 = \frac{1}{180}c_0 + \frac{1}{180}c_1, \dots$$

$$y = c_0 + c_1 x - \frac{1}{6}c_0 x^3 - \frac{1}{12}c_1 x^4 + \frac{1}{120}c_0 x^5 + \frac{1}{180}c_0 x^6 + \frac{1}{180}c_1 x^6 + \dots$$

$$= c_0 \left[1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \dots \right]$$

$$+ c_1 \left[x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \dots \right]$$

Since, no singular points for the DE $\Rightarrow R = \text{radius of conv} = \infty$

$$\triangle y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 + \dots \quad |x| < \infty$$

$$\triangle y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \dots \quad |x| < \infty$$

(Q6) $x_0 = 0$ is a regular singular point of the DE

$$4x^2 y'' + 4xy' + (4x^2 - 1)y = 0$$

Find the indicial roots of the singularity.

(a) $r_1 = 1, r_2 = -1$

(b) $r_1 = \frac{1}{2}, r_2 = -\frac{1}{2}$

(c) $r_1 = \frac{1}{4}, r_2 = -\frac{1}{4}$

(d) $r_1 = 0, r_2 = -\frac{1}{2}$

$$y'' + \frac{1}{x} y' + \left(1 - \frac{1}{4x^2}\right) y = 0$$

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \frac{1}{4x^2}$$

$$f(x) = x P(x) = 1, \quad g(x) = x^2 Q(x) = x^2 - \frac{1}{4}$$

$$a_0 = f(0) = 1, \quad b_0 = g(0) = -\frac{1}{4}$$

indicial equation: $r(r-1) + a_0 r + b_0 = 0$

$$r(r-1) + r - \frac{1}{4} = 0$$

$$4r(r-1) + 4r - 1 = 0$$

$$4r^2 - 4r + 4r - 1 = 0$$

$$4r^2 - 1 = 0$$

$$(2r-1)(2r+1) = 0$$

$$r = \frac{1}{2}, -\frac{1}{2}$$

$$r_1 = \frac{1}{2}, \quad r_2 = -\frac{1}{2}$$

(Q7) The form of two linearly independent solutions of the DE

$$xy'' + (1-x)y' - y = 0$$

(r_1, r_2 are the indicial roots)

(a) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}, b_0 \neq 0$

(b) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, y_2 = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_1}, b_0 \neq 0$

(c) $y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0, y_2 = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}$

(d) none of the above

$$y'' + \left(\frac{1}{x} - 1\right)y' - \frac{1}{x}y = 0$$

$$P(x) = \frac{1}{x} - 1, \quad Q(x) = -\frac{1}{x}$$

$$P(x) = 1 - x, \quad Q(x) = -x$$

$$a_0 = 1, \quad b_0 = 0$$

$$r(r-1) + r = 0 \Rightarrow r^2 - r + r = 0$$

$$\Rightarrow r^2 = 0$$

$$r_1 = r_2 = 0$$

\Rightarrow equal indicial roots

(Q8) Without solving classify each of the following equations as to: (circle the appropriate word)

(a) separable

(b) Exact

(c) Bernolli

(I)	$(x+y)^2 dx + (2xy + x^2 - 1) dy = 0$	separable	exact	Bernolli
(II)	$y' + \frac{y}{x} = xy^2$	separable	exact	Bernolli
(III)	$y' = \frac{y+1}{x}$	separable	exact	Bernolli
(IV)	$y' = (x+1)^2$	separable	exact	Bernolli

3 each

(Q9) Transform the system

$$3y_1'' - 5y_2' + t^2 = 0$$

$$7y_2'' + 5y_1' = e^t$$

Into a first-order system

$$X'(t) = AX(t) + F(t).$$

$$x_1 = y_1 \Rightarrow x_1' = y_1' = x_2$$

$$x_2 = y_1' \Rightarrow x_2' = y_1'' = \frac{5}{3}y_2' + \frac{1}{3}t^2 = \frac{5}{3}x_4 - \frac{1}{3}t^2$$

$$x_3 = y_2 \Rightarrow x_3' = y_2' = x_4$$

$$x_4 = y_2' \Rightarrow x_4' = y_2'' = -\frac{5}{7}y_1' + \frac{1}{7}e^t = -\frac{5}{7}x_2 + \frac{1}{7}e^t$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{5}{7} & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ -\frac{1}{3}t^2 \\ 0 \\ \frac{1}{7}e^t \end{bmatrix}$$

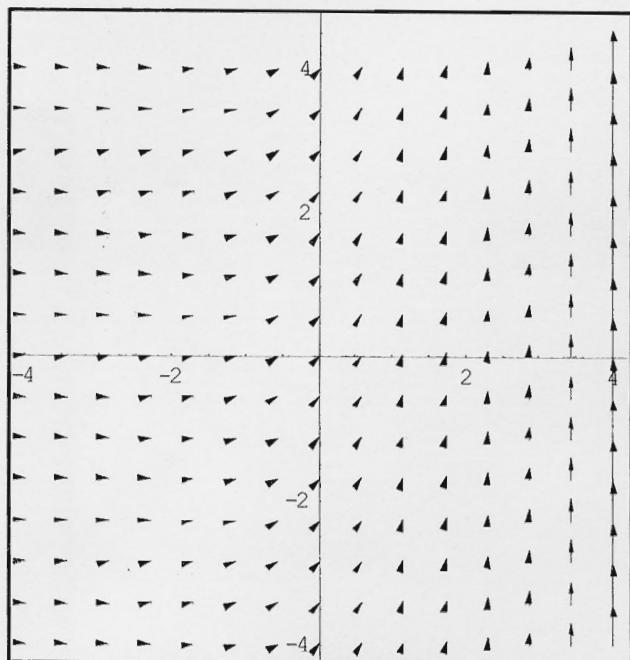
$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{5}{7} & 0 & 0 \end{bmatrix}$	$F(t) = \begin{bmatrix} 0 \\ -\frac{1}{3}t^2 \\ 0 \\ \frac{1}{7}e^t \end{bmatrix}$
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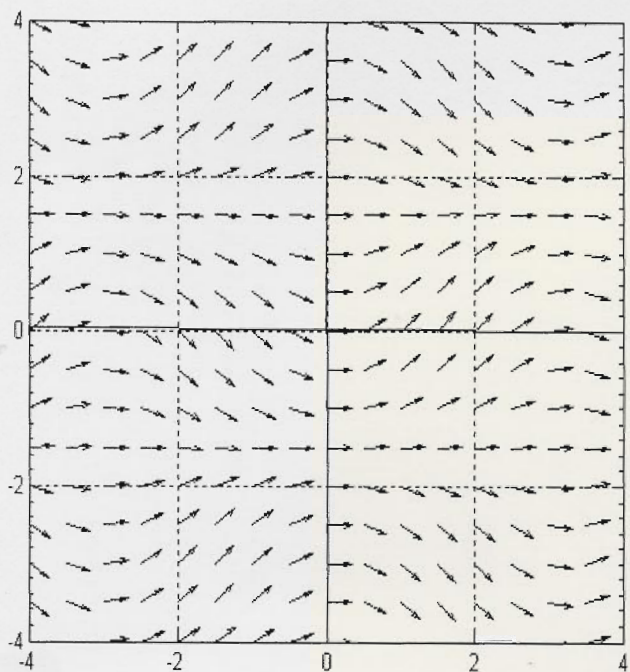
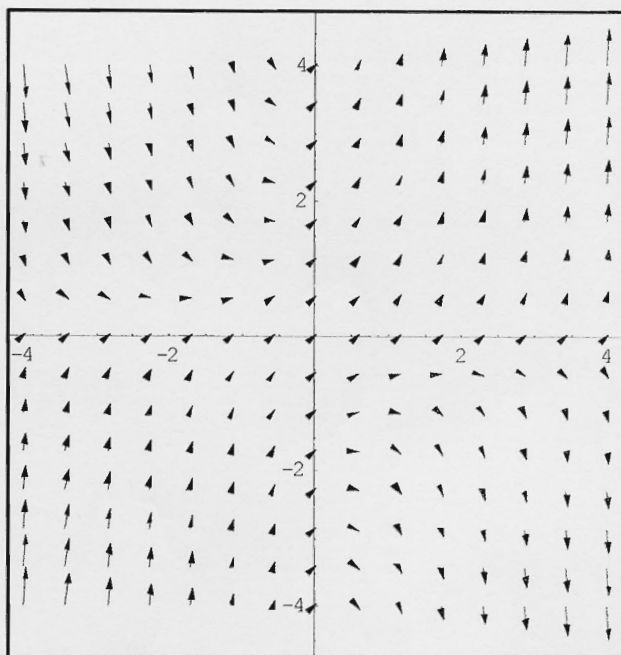
5

(Q10) The computer-generated directional field of $y' = x^2 y^2$ is :

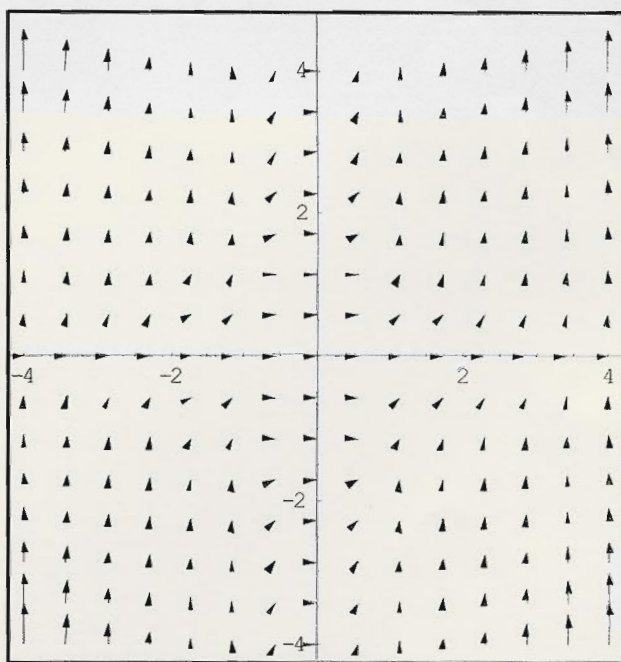
(a)



(b)



(c)



(d)

(Q11) State the order of the given differential equation. Determine whether the equation is ODE or PDE. Determine whether the equation is linear or nonlinear.

	Equation	Type	Linearity	Order
1	A simple model for the shape of a tsunami, or tidal wave, is given by $\frac{dW}{dx} = W\sqrt{4 - 2W}$	ODE PDE	linear nonlinear	1
2	The model of the free pendulum given by $\frac{d^2\theta}{dt^2} + 2\lambda\frac{d\theta}{dt} + \omega^2 \sin\theta = 0$ Where λ, ω are constants.	ODE PDE	linear nonlinear	2
3	The steady state heat conduction equation $-\frac{\partial}{\partial x}\left(k_1\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_2\frac{\partial u}{\partial y}\right) = -\frac{\partial\rho}{\partial x}$ Where k_1 and k_2 are constants.	ODE PDE	linear nonlinear	2
4	The equation of a vibrating membrane $\frac{\partial^2 u}{\partial t^2} - a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$ Where a is a constant.	ODE PDE	linear nonlinear	2
5	The current $i(t)$ in an LRC series circuit satisfies $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)$ Where L, R, C are constants.	ODE PDE	linear nonlinear	2

1 point each

(Q12) The following IVP

$$X' = \begin{bmatrix} \frac{1}{\sqrt{10-t}} & 5 \\ t^2 & \sin t \end{bmatrix} X + \begin{bmatrix} \frac{t}{\sin t} \\ \ln(t+50) \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

has a unique solution on the interval

(a) $(0,10)$

(b) $(0,+\infty)$

(c) $(0,4)$

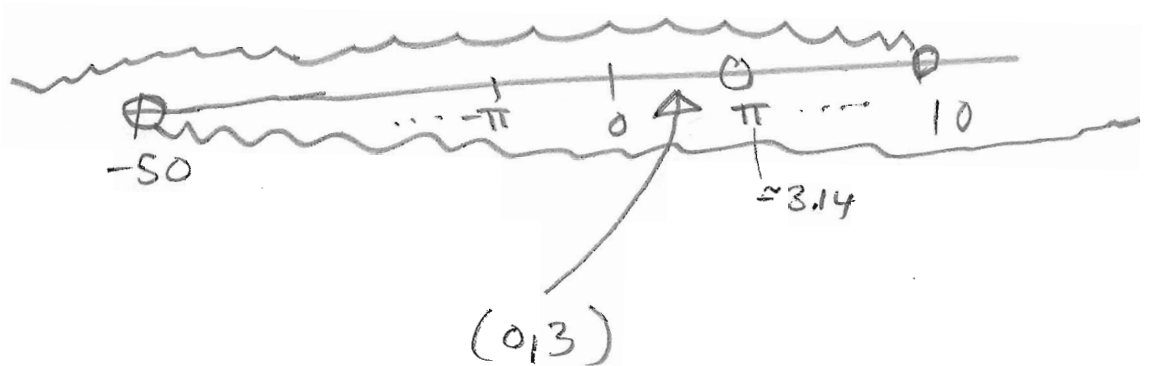
(d) $(-10,10)$

(e) $(0,3)$

$$10-t > 0$$

$$t+50 > 0$$

$$\sin t \neq 0 \Rightarrow t = \pm \text{multiple } \pi$$



(Q13) The general solution for the DE

$$y'' - y = 0$$

(a) $y = c_1 e^{2x} + c_2 e^{-2x}$

(b) $y = c_1 \sin x + c_2 \cos x$

(c) $y = c_1 \sinh x + c_2 \cosh x$

$$y'' - y = 0$$

$$(D^2 - 1)y = 0$$

Ch. eq: $m^2 - 1 = 0$

roots: $m = +1, -1$

sols: $y_1 = e^x, y_2 = e^{-x}$

or sols: $y_1 = \sinh x, y_2 = \cosh x$

(Q14) Given that $y_1 = x^{-1}$, $y_2 = x^4$ are solutions for the DE

$$x^2 y'' - 2xy' - 4y = 0 \quad (*)$$

and $y_p = -1$ is a particular solution for

$$x^2 y'' - 2xy' - 4y = 4 \quad (**)$$

Then the solution for the IVP

$$x^2 y'' - 2xy' - 4y = 4$$

$$y(1) = 2, y'(1) = 2$$

(circle the correct answer)

(a) $y = 2x^{-1} + x^4$

(b) $y = 2x^{-1} + x^4 - 1$

(c) $y = x^{-1} + 2x^4 - 1$

general sol for (**)

$$y = c_1 x^{-1} + c_2 x^4 + (-1)$$

$$y' = -c_1 x^{-2} + 4c_2 x^3$$

$$2 = y(1) = c_1 + c_2 - 1$$

$$2 = y'(1) = -c_1 + 4c_2$$

$$\Rightarrow \begin{cases} c_1 + c_2 = 3 \\ -c_1 + 4c_2 = 2 \end{cases}$$

$$\hline 5c_2 = 5 \Rightarrow c_2 = 1$$

$$\Rightarrow c_1 = 2$$

Sol for IVP:

$$\boxed{y = 2x^{-1} + x^4 - 1}$$

(Q15) True or False

(a)	The point $x = 2$ is a regular singular point of the DE $(x^2 - 4)y'' + 3(x - 2)y' + 5y = 0$	T	F
(b)	The differential operator $[D^2 - 2D + 2]^2$ annihilates the function $xe^x(\sin x + \cos x)$	T	F
(c)	$x = -2$ is an ordinary point of the DE $3x(x + 2)y'' + (x - 2)y' - y = 0$	T	F
(d)	$(\frac{1}{e^{i\beta}})^2 = \cos(2\beta) - i\sin(2\beta)$	T	F
(e)	The DE: $x^2y'' - xy' + 2y = 0$ is a Cauchy-Euler equation.	T	F
(f)	If $\Phi(t)$ is a fundamental matrix of the system $X' = AX$, then $\Phi'(t) = A\Phi(t)$.	T	F
(g)	We can not find two linearly independent solutions of the DE $xy'' + y = 0$ in the form of a power series centered at $x = 2$.	T	F
(h)	$\left. \begin{array}{l} x^2y'' - xy' + 2y = 0 \\ y(0) = 1, y(1) = 0 \end{array} \right\}$ is a Boundary Value Problem (BVP)	T	F
(i)	A constant multiple of a solution of a homogeneous linear differential equation is also a solution.	T	F
(j)	A set of functions is linearly dependent if at least one function can be expressed as a linear combination of the remaining functions.	T	F
(k)	If $y_1 = 1, y_2 = x^2$ then $W(y_1, y_2) = -2x$	T	F

2 point each

(Bonus 1) Consider the DE:

$$(x+2)y'' + \frac{1}{x-5}y' + \frac{x-2}{(x^2+4)(x^2-6x+14)}y = 0$$

Without solving the DE, find a lower bound for the radius of convergence of power series solutions about the point $x = 2$. The power series solutions centered at $x = 2$ will converge at least for $|x - 2| < R$. Then $R =$

Show all your work then Circle your answer

(a) 3

(b) 2

(c) $2\sqrt{2}$

(d) $2\sqrt{3}$

(e) 4

(f) $2\sqrt{1.5}$

Singular points
 $x+2=0$ or $x-5=0$, $x^2-4=0$, $x^2-6x+14=0$

$$x^2 - 6x + 14 = 0 \Rightarrow x = \frac{6 \pm \sqrt{36 - 56}}{2} = \frac{6 \pm \sqrt{-20}}{2} = 3 \pm 2\sqrt{5}i$$

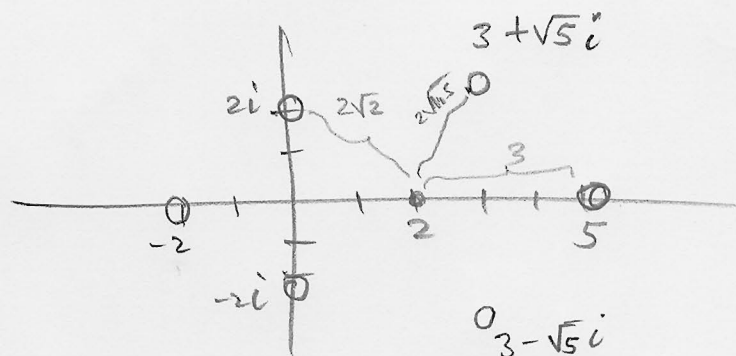
Singular points $x = -2, 5, 2i, -2i$

$$x = 3 \pm \sqrt{5}i$$

$x = 2$ is an ordinary point $\Rightarrow y = \sum_{n=0}^{\infty} c_n (x-2)^n$

$$|x-2| < R$$

$R =$ closest distance to singular point



$$\text{dist}(2, 3 + \sqrt{5}i) = \sqrt{(3-2)^2 + (\sqrt{5})^2} = \sqrt{1+5} = \sqrt{6} = \sqrt{\frac{4 \times 6}{4}} = 2\sqrt{1.5}$$