King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

CODE 004 **Math 102** CODE 004

Exam I

103

Tuesday, July 12, 2011

Net Time Allowed: 120 minutes

Name: ______________________________

ID: ____________________ Sec: ____________________.

Check that this exam has 20 questions.

Important Instructions:

1. All types of calculators, pagers or mobile phones are NOT allowed during the examination.
2. Use HB 2.5 pencils only.
3. Use a good eraser. DO NOT use the erasers attached to the pencil.
4. Write your name, ID number and Section number on the examination paper and in the upper left corner of the answer sheet.
5. When bubbling your ID number and Section number, be sure that the bubbles match with the numbers that you write.
6. The Test Code Number is already bubbled in your answer sheet. Make sure that it is the same as that printed on your question paper.
7. When bubbling, make sure that the bubbled space is fully covered.
8. When erasing a bubble, make sure that you do not leave any trace of penciling.

1. $\int_1^2 \frac{e^{3/x}}{x^2}\,dx = I$. Let $u = \frac{3}{x} \Rightarrow du = -\frac{3}{x^2}dx$. So $\frac{dx}{x^2} = -\frac{1}{3}du$

When $x=1$, $u=3$ and when $x=2$, $u=\frac{3}{2}$

$$I = \int_3^{3/2} -\frac{1}{3} e^u\,du = \frac{1}{3}\int_{2/3}^{3} e^u\,du$$

$$= \frac{1}{3}\left[e^u\right]_{2/3}^{3} = \frac{1}{3}\left(e^3 - e^{3/2}\right).$$

(a) $3\left(e^3 - e^{3/2}\right)$

(b) $-3\left(e^3 - e^{3/2}\right)$

(c) $\frac{1}{3}\left(e^3 + e^{3/2}\right)$

(d) $\frac{-1}{3}\left(e^3 - e^{3/2}\right)$

$\boxed{\text{(e) } \frac{1}{3}\left(e^3 - e^{3/2}\right)}$

2. An object moves along a horizontal line with velocity $v(t) = t^2 - 8t + 7$ (meter/second). The total distance traveled by the particle from $t = 0$ to $t = 4$ is

$V(t) \geq 0$ if $0 \leq t \leq 1$

and $V(t) < 0$ if $1 < t \leq 4$. Thus the total distance d is

$$d = \int_0^1 V(t)\,dt + \int_1^4 [-V(t)]\,dt = \int_0^1 (t^2 - 8t + 7)\,dt$$

$$+ \int_1^4 (8t - t^2 - 7)\,dt = \left[\frac{t^3}{3} - 4t^2 + 7t\right]_0^1 + \left[4t^2 - \frac{t^3}{3} - 7t\right]_1^4$$

$$= \frac{1}{3} - 4 + 7 + 64 - \frac{64}{3} - 28 - 4 + \frac{1}{3} + 7$$

$$= \frac{64}{3}$$

(a) $\frac{44}{3}$

$\boxed{\text{(b) } \frac{64}{3}}$

(c) $\frac{54}{3}$

(d) $\frac{34}{3}$

(e) $\frac{128}{3}$

3. The value of the definite integral $\int_0^{3/2} \sqrt{9-4x^2}\, dx$ is
(Hint: you may use areas)

$\sqrt{9-4x^2} = 3\sqrt{1-\left(\frac{2x}{3}\right)^2}$.

Let $u = \frac{2}{3}x \Rightarrow du = \frac{2}{3}dx$. When $x=0, u=0$ and when $x=\frac{3}{2}$, $u=1$. Therefore

$\int_0^{3/2} \sqrt{9-4x^2}\, dx = \frac{3}{2}\int_0^1 3\sqrt{1-u^2}\, du = \frac{9}{2}\int_0^1 \sqrt{1-u^2}\, du$

But $\int_0^1 \sqrt{1-u^2}\, du$ is the area of the region in the 1^{st} quadrant bounded by $y=\sqrt{1-u^2}$ i.e $y^2+u^2=1$ i.e the circle of center the origin and radius 1.

Therefore: $\int_0^{3/2} \sqrt{9-4x^2}\, dx = \frac{9}{2}\cdot\frac{\pi}{4} = \frac{9\pi}{8}$

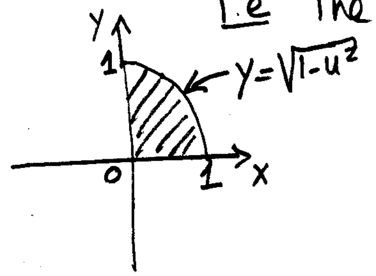
(a) $3\pi/8$

(b) $9\pi/16$

(c) $4\pi/4$

$\boxed{\text{(d) } 9\pi/8}$

(e) $\pi/8$



4. If $f(x)$ is an even, continuous function for which $\int_{-2}^{2} f(x)dx = 6$ and if also $\int_0^4 f(x)dx = 10$, then $\int_2^4 f(x)dx =$

$\int_{-2}^{2} f(x)dx \underset{\substack{\uparrow \\ \text{because } f \\ \text{is even}}}{=\!=} 2\int_0^2 f(x)dx = 6$. Thus $\int_0^2 f(x)dx = 3$.

Now: $\int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx$. Hence

$\int_2^4 f(x)dx = \int_0^4 f(x)dx - \int_0^2 f(x)dx = 10 - 3 = 7$

(a) -4

$\boxed{\text{(b) } 7}$

(c) 16

(d) 14

(e) 4

5. The area of the region bounded by the curves $y^2 = 2x + 1$ and $x + y = 1$ is equal to

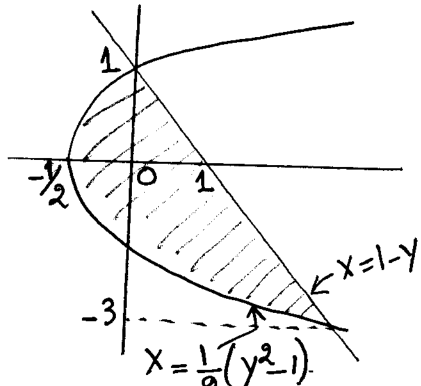
(a) 17/3

$\boxed{\text{(b) } 16/3}$

(c) 15/4

(d) 4

(e) 13/2



The y-coordinates of the intersection points:
$1-y=\frac{1}{2}(y^2-1) \Leftrightarrow y^2+2y-3=0$
$\Leftrightarrow y=1$ or -3.

$A=\int_{-3}^{1}\left[(1-y)-\frac{1}{2}(y^2-1)\right]dy$

$=\int_{-3}^{1}\left(\frac{3}{2}-y-\frac{1}{2}y^2\right)dy$

$=\frac{3}{2}y-\frac{y^2}{2}-\frac{1}{6}y^3\Big|_{-3}^{1}=\frac{16}{3}$

6. Evalute the integral $\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)dx = I$

Let $u=\sin\left(\frac{x}{3}\right)$; $du=\frac{1}{3}\cos\left(\frac{x}{3}\right)dx$.

$I=\int 3u^5\,du=\frac{3}{6}u^6+C$

$=\frac{1}{2}\sin^6\left(\frac{x}{3}\right)+C$

(a) $\frac{1}{6}\sin^6\left(\frac{x}{3}\right)+C$

(b) $\frac{1}{2}\cos^6\left(\frac{x}{3}\right)+C$

$\boxed{\text{(c) } \frac{1}{2}\sin^6\left(\frac{x}{3}\right)+C}$

(d) $2\sin^6\left(\frac{x}{3}\right)+C$

(e) $\frac{1}{6}\cos^6\left(\frac{x}{3}\right)+C$

7. Evaluate $I = \lim_{t \to 0^+} \frac{d}{dt} \int_1^{\sqrt{t}} \frac{\sin x^2}{x}\, dx$

$$\frac{d}{dt}\int_1^{\sqrt{t}} \frac{\sin x^2}{x}\,dx = \frac{1}{2\sqrt{t}}\,\frac{\sin\left(\sqrt{t}^{\,2}\right)}{\sqrt{t}} = \frac{1}{2}\,\frac{\sin t}{t}$$

Since $\lim_{t \to 0^+} \frac{\sin t}{t} = 1$, we have that $I = \frac{1}{2}$

(a) $I = 0$

(b) $I = 2$

(c) $I = 1/2$

(d) $I = 5/2$

(e) The limit does not exist

8. The limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2}{n} + \frac{3k}{n^2}\right) = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{2}{n}\left(1 + \frac{3}{2}\,\frac{k}{n}\right) = \lim_{n\to+\infty} \frac{2}{n} \sum_{k=1}^{n} \left(1 + \frac{3}{4} k \frac{2}{n}\right)$$

can be written as a definite integral on the interval $[0, 2]$ as $= \int_0^2 \left(1 + \frac{3}{4}x\right)dx$

Here: • $\Delta x = \frac{2}{n}$.
• The initial point is 0
• The endpoint is 2
• The Riemann sum is taken using the right endpoints.

(a) $\int_0^2 \left(2 + \frac{3}{2}x\right) dx$

(b) $\int_0^2 \left(1 + \frac{3}{2}x\right) dx$

(c) $\int_0^2 (2 + 3x)\, dx$

(d) $\int_0^2 \left(2 + \frac{3}{4}x\right) dx$

(e) $\int_0^2 \left(1 + \frac{3}{4}x\right) dx$

9. If $\int_0^2 g(x)dx = 6$, then $\int_2^3 3g(2t-4)dt =$

Let $x = 2t - 4$. $dx = 2dt$. When $t = 2$, $x = 0$ and when $t = 3$, $x = 2$. Thus:

$$\int_2^3 3g(2t-4)dt = \int_0^2 \frac{3}{2} g(x)dx = \frac{3}{2}\int_0^2 g(x)dx = 9$$

(a) 6

(b) 18

(c) 12

(d) 9/2

$\boxed{\text{(e) } 9}$

10. Evaluate $\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx$. Let $u = e^x$, $du = e^x dx$. When $x = 0$, $u = 1$ and when $x = \ln\sqrt{3}$, $u = \sqrt{3}$. Therefore:

$$\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}dx = \int_1^{\sqrt{3}} \frac{du}{1+u^2} = \tan^{-1}u\Big|_1^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{3}$

$\boxed{\text{(c) } \frac{\pi}{12}}$

(d) $\ln 2$

(e) $2\ln 2$

11. Evaluate the integral $\int \frac{dx}{2+(x-1)^2} = \frac{1}{2}\int \frac{dx}{1+\left(\frac{x-1}{\sqrt{2}}\right)^2}$.

Let $u = \frac{x-1}{\sqrt{2}}$, $du = \frac{1}{\sqrt{2}}dx$. Thus

$$\int \frac{dx}{2+(x-1)^2} = \frac{\sqrt{2}}{2}\int \frac{du}{1+u^2} = \frac{\sqrt{2}}{2}\tan^{-1}u + C$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C$$

(a) $\boxed{\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C}$

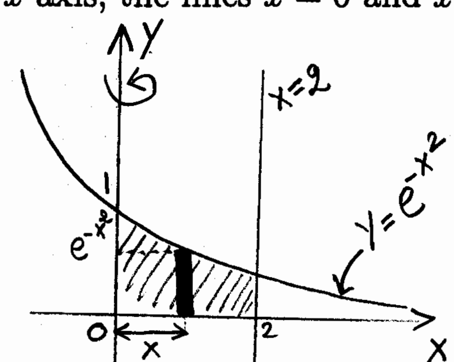
(b) $\frac{1}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

(c) $\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{2}\right)+C$

(d) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$

(e) $\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

12. What is the volume of a solid of revolution generated by rotating around the y-axis the region enclosed by the graph of $y = e^{-x^2}$, the x-axis, the lines $x = 0$ and $x = 2$



(a) $\boxed{\pi(1-e^{-4})}$

(b) $\pi(e^4-1)$

(c) $\pi(1-e^{-2})$

(d) πe^4

(e) πe^{-4}

$$V = \int_0^2 e^{-x^2} 2\pi x\, dx = 2\pi\int_0^2 xe^{-x^2}dx = 2\pi\left[-\frac{1}{2}e^{-x^2}\right]_0^2$$

$$= \pi(1-e^{-4})$$

13. $\int \frac{x-1}{\sqrt{1-x^2}}\,dx = \int\left(\frac{x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right)dx = -\sqrt{1-x^2} - \sin^{-1}x + C$

(a) $2\sqrt{1-x^2} - \sin^{-1} x + C$

(b) $\boxed{-\sqrt{1-x^2} - \sin^{-1} x + C}$

(c) $\sqrt{1-x^2} - \sin^{-1} x + C$

(d) $-2\sqrt{1-x^2} - \sin^{-1} x + C$

(e) $-(1-x^2) - \sin^{-1} x + C$

14. Evaluate $\int_1^e \frac{\ln x}{x} e^{(\ln x)^2}\,dx = I$

Let $u = (\ln x)^2$, $du = \frac{2}{x}\ln x\,dx$
when $x=1$, $u=0$
and when $x=e$, $u=1$.
$I = \frac{1}{2}\int_0^1 e^u\,du = \frac{1}{2}(e-1)$

(a) $\frac{1}{2}(1-e)$

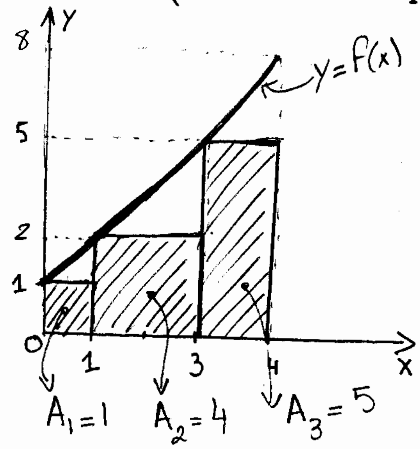
(b) $e-1$

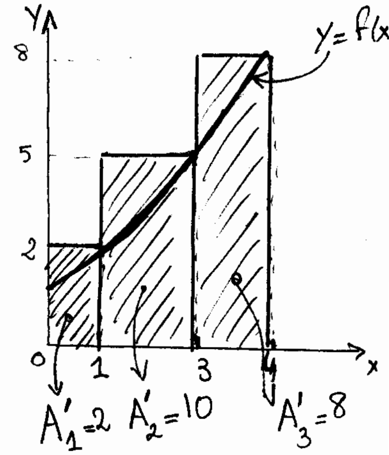
(c) $\boxed{\frac{1}{2}(e-1)}$

(d) $\frac{1}{2}e$

(e) e^2

15. Let $f(x)$ be a function with properties that $f'(x) > 0$ for all x, and that $f(0) = 1, f(1) = 2, f(3) = 5$ and $f(4) = 8$. Which of the following **must** be true? (Hint: Draw a picture!)





(a) $26 \le \int_0^4 f(x)dx \le 33$

(b) $7 \le \int_0^4 f(x)dx \le 9$

(c) $\boxed{10 \le \int_0^4 f(x)dx \le 20}$

(d) $21 \le \int_0^4 f(x)dx \le 25$

(e) $1 \le \int_0^4 f(x)dx \le 6$

$$\underbrace{A_1 + A_2 + A_3}_{10} \leq \int_0^4 f(x)\,dx \leq \underbrace{A_1' + A_2' + A_3'}_{20}$$

16. Suppose $\int_0^x f(t)dt + 2\sin x = 4x$. What is the value of $f(\pi)$?

$$\frac{d}{dx}\left(\int_0^x f(t)dt + 2\sin x\right) = \frac{d}{dx}(4x)$$

$f(x) + 2\cos x = 4$. Thus

$f(\pi) + 2\cos\pi = 4$. Hence $f(\pi) = 6$

(a) 6π

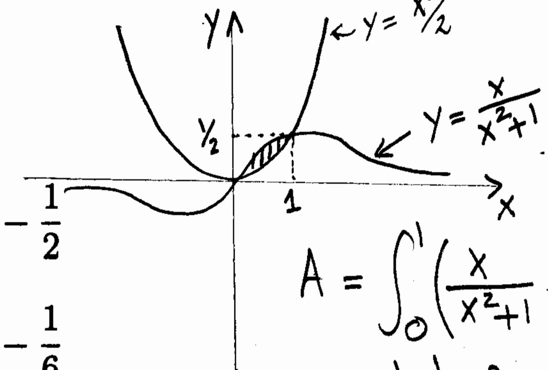
(b) $\boxed{6}$

(c) 4

(d) 2

(e) 4π

17. Calculate the area bounded by the curves $y = \dfrac{x}{x^2+1}$ and $y = \dfrac{x^2}{2}$ is



$$A = \int_0^1 \left(\frac{x}{x^2+1} - \frac{x^2}{2} \right) dx = \frac{1}{2}\ln(x^2+1) - \frac{x^3}{6} \Big|_0^1$$
$$= \frac{1}{2}\ln 2 - \frac{1}{6}$$

(a) $\frac{1}{6}\ln 6 - \frac{1}{2}$

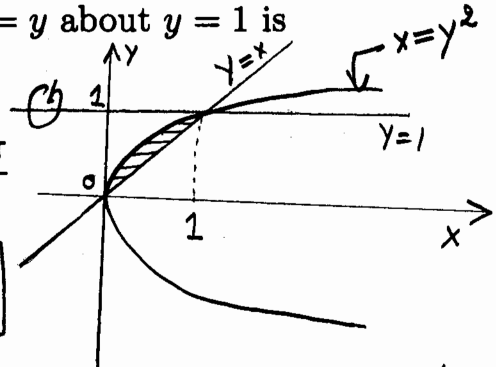
(b) $\frac{1}{2}\ln 6 - \frac{1}{6}$

(c) $\boxed{\frac{1}{2}\ln 2 - \frac{1}{6}}$

(d) $\frac{1}{6}\ln 2 - \frac{1}{2}$

(e) 1

18. Using the shell method, one can show that the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = y$ about $y = 1$ is



$$V = \int_0^1 2\pi(1-y)(y-y^2)\,dy$$
$$= 2\pi \int_0^1 (y - y^2 - y^2 + y^3)\,dy$$
$$= 2\pi \left[\frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4} \right]_0^1$$
$$= 2\pi \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{\pi}{6}$$

(a) $\frac{3\pi}{4}$

(b) $\boxed{\frac{\pi}{6}}$

(c) $\frac{2\pi}{3}$

(d) $\frac{\pi}{12}$

(e) $\frac{5\pi}{6}$

19. The Riemann sum for $f(x) = x+2$ which results from partitioning the interval $[-1, 1]$ into n equal-width subintervals, and using the left endpoint of each subinterval as a sample point is

$$R_n(x) = \sum_{i=0}^{n-1} f(x_i)\Delta x = \frac{2}{n}\sum_{i=0}^{n-1} f\left(-1+i\frac{2}{n}\right)$$
$$= \frac{2}{n}\sum_{i=0}^{n-1}\left(-1+i\frac{2}{n}+2\right) = \frac{2}{n}\sum_{i=0}^{n-1}\left(1+i\frac{2}{n}\right)$$
$$= \frac{2}{n}\left(n+\frac{2}{n}\sum_{i=0}^{n-1} i\right) = \frac{2}{n}\left(n+\frac{2}{n}\frac{n(n-1)}{2}\right)$$
$$= 2+\frac{2}{n}(n-1) = 4-\frac{2}{n}$$

(a) $4 - 3/n$

(b) $4 - 4/n$

(c) $4 - 2/n$

(d) $4 - 2/n^2$

(e) $4 + 2/n^2$

20. The base of a solid is a semi-circular disk $\{(x, y) | x^2 + y^2 \leq 1, \quad x \geq 0\}$. Cross-sections perpendicular to the x-axis are squares with two of their vertices on the semi-circle. Compute the volume of the solid.

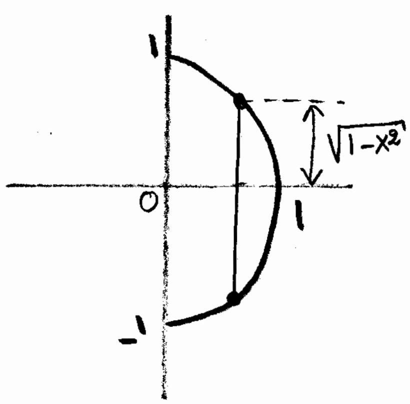
Area of one section is:

$$A(x) = \left(2\sqrt{1-x^2}\right)^2 = 4(1-x^2)$$

with $0 \leqslant x \leqslant 1$

Therefore:

$$V = \int_0^1 A(x)\,dx = 4\int_0^1 (1-x^2)\,dx$$
$$= 4\left[x - \frac{x^3}{3}\right]_0^1 = 4\left(1-\frac{1}{3}\right) = \frac{8}{3}$$



(a) $\frac{2\pi}{3}$

(b) $\frac{\pi^2}{4}$

(c) π^2

(d) 4

(e) 8/3