King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences
Math 101
Final Exam
Semester I, 2001–2002 (011)
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Serial No:	KEY

Form (1)

Question #		Points
1		10
2		27
3		34
4		27
5		27
6		9
7-23		8 each
24		3 each
Total:		300

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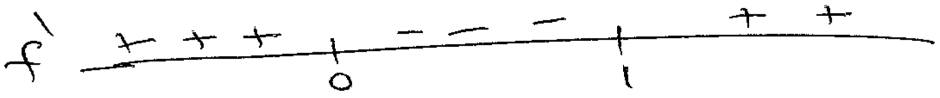
1. Let $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$

(a) Find the interval on which f is decreasing

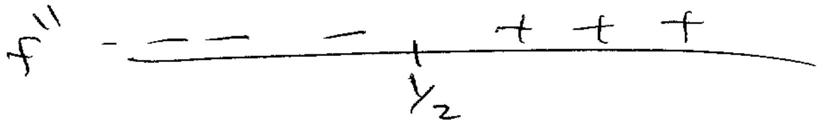
$f'(x) = x^2 - x = x(x-1)$



$\Rightarrow$ f is decreasing on $(0,1)$

(b) Find the open interval on which f is concave up.

$f''(x) = 2x - 1$



$\Rightarrow$ f'' is concave up on $(\frac{1}{2}, +\infty)$

2. The equation $x^3 - x^2 - 2x + 1 = 0$ has one real solution for $0 < x < 1$. Approximate it by Newton's Method. If $x_1 = 1$, then find x_6.

$$f(x) = x^3 - x^2 - 2x + 1 \quad , \quad f'(x) = 3x^2 - 2x - 2$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{with } x_1 = 1$$

n	x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	-1.0000	-1.0000	1.0000	0
2	0.0000	1.0000	-2.0000	-0.5000	0.5000
3	0.5000	-0.1250	-2.2500	-0.0555	0.4444
4	0.4444	0.00137	-2.29629	-0.000597	0.445041816
5	0.445041816	0.000000119	-2.29589	-0.0000000519	0.4450418679
6	0.4450418679	$\Longrightarrow x_6 = 0.4450418679$			

Now, the real solution is $\cong 0.4450418$

3. Find the extreme values for $f(x) = x^{4/3} - 3x^{1/3}$ on the interval $[-1, 8]$ and determine where those values occur.

$$f'(x) = \frac{4}{3}X^{\frac{1}{3}} - X^{-2/3} = \frac{4}{3}X^{\frac{1}{3}} - \frac{1}{X^{2/3}} = \frac{4x-3}{3X^{2/3}}$$

$\Rightarrow$ $X = \frac{3}{4}, 0$ are critical points

check: -1, 0, $\frac{3}{4}$, 8

$f(-1) = 4$, $f(0)=0$, -2.04, $f(8) = 10$

$f\,\cancel{f}(\frac{3}{4})=$

* f has absoulte max at $x=8$
and abs. max $= 10$

* f has abs. min at $X = \frac{3}{4}$
and abs. min $= -2.04$

4. Let $f(x) = x^{2/3}(x+5)$

(a) Find all relative max. and all relative min.

$$f'(x) = \frac{2}{3}x^{-1/3}\cdot(x+5) + x^{2/3}\cdot(1)$$

$$= \frac{2(x+5)+3x}{3x^{1/3}} = \frac{5x+10}{3x^{1/3}} = \frac{5(x+2)}{3x^{1/3}}$$

$x = 0, -2$ are critical points

		-2		0	
$5(x+2)$	$- - -$		$+ +$		$+ +$
$3x^{\frac{1}{3}}$	$- -$		$- -$		$+ +$
f'	$+ + +$		$- -$		$+ +$

$\Rightarrow \boxed{x=-2}$ (f has relative max.)

$\Rightarrow \boxed{x=0}$ (f has relative min)

(b) Find all inflection points

$$f''(x) = \frac{15x^{1/3} - 5x^{-2/3}(x+2)}{9x^{2/3}} = \frac{15x - 5(x+2)}{9x^{4/3}}$$

$$= \frac{10x-10}{9x^{4/3}} = \frac{10}{9}\cdot\frac{(x-1)}{x^{4/3}}$$

$x-1$	$-$	0	$-$		$+$
$x^{4/3}$	$+$		$+$		$+$
f''	$-$		$-$		$+$

$\Rightarrow$ f has inflection point at $\boxed{x=1}$

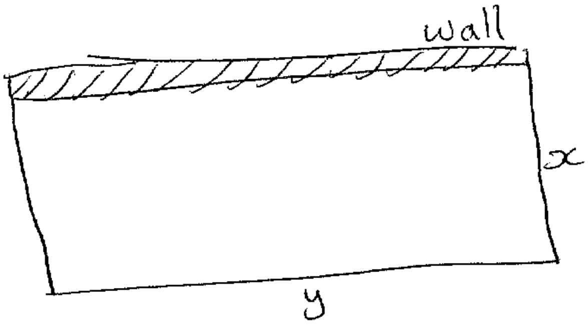
(c) Find all cusp which may or may not exist.

$x=0$ is a point of non-diff.

and $\lim_{x\to 0^+} f'(x) = +\infty$ and $\lim_{x\to 0^-} f'(x) = -\infty$

Hence f has cusp at $\boxed{x=0}$

5. A rectangular field is to be bounded by a fence on three sides and by a wall on the fourth one. Find the dimensions of the field with maximum area that can be enclosed with 1000 meters of fence ?



It is a maximization problem. we need to maximize the area of the rectangle.

$A = x \cdot y$ ——————— (1)

we have ; $2x + y = 1000$

$\Rightarrow y = 1000 - 2x$ ——— (2)

from (1) and (2),

$A(x) = x(1000 - 2x) = 1000x - 2x^2$

Also $x \in [0, 500]$

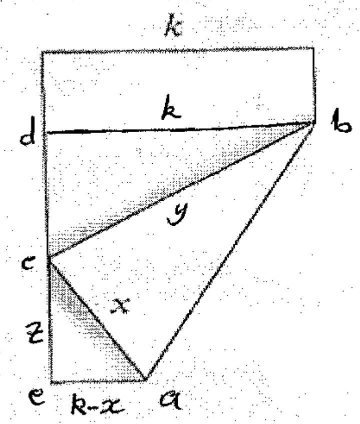
$A'(x) = 1000 - 4x \Rightarrow x = 250$ is a critical pt.

check , $x = 0 , 250 , 500$

$A(0) = 0 , A(250) = 125000 , A(500) = 0$

Hence, the dimensions of the field with maximum area are $X = 250 , y = 500$

6. The lower corner of a page of width k is folded over so as just to reach the inner edge of the page (see Figure). Find the width of the part folded over (the value of x) when the area of the triangle floded over is a minimum.



Figure

Answer:

$x = \frac{2k}{3}$

for example, for A4 size paper (21 cm × 29.7 cm)

$x = 14$ cm

It is a minimization problem. we need to minimize the area of Δabc

Now, $A = \frac{1}{2}xy$ ——— (1)

The two triangle Δabc and Δbcd are similar (we have $\hat{bdc} = \hat{cea} = 90°$ and $\hat{ace} = \hat{cbd}$).

So, $\frac{y}{x} = \frac{k}{z}$ ——— (2)

In the right triangle Δace, we have: $z^2 = x^2 - (k-x)^2$

So, $z^2 = x^2 - k^2 - x^2 + 2kx = 2kx - k^2 \Rightarrow z = \sqrt{k(2x-k)}$ —(3)

(2) & (3) give: $y = \frac{kx}{\sqrt{k(2x-k)}}$ ——— (4)

(1) & (4) give: $\boxed{A(x) = \frac{kx^2}{2\sqrt{k(2x-k)}} \quad \text{and } x \in [\tfrac{1}{2}k, k]}$

$A'(x) = \frac{k(2k-3x)x}{2(2x-k)^{3/2}} \Rightarrow x = \frac{2k}{3}$ is a critical point

$A''(x) = \frac{k(3x^2-4kx+2k^2)}{2(2x-k)^{5/2}} \Rightarrow A''(\frac{2k}{3}) = 3\sqrt{3}\sqrt{k} > 0$

$\Rightarrow x = \frac{2k}{3}$ is the only relative extrem and is relative min $\Rightarrow f$ has Abs. min at $x = 2k/3$

7. If $f(x) = x^4 - x$ on $[-1, 1]$, find the value c that satisfies the Mean value Theorem.

(a) 1

[✓ (b) 0]

(c) 2

(d) 3

(e) 4

$$f'(x) = 4x^3 - 1$$

$$f'(c) = \frac{f(1) - f(-1)}{1-(-1)} = \frac{0-2}{2} = -1$$

$$4c^3 - 1 = -1 \Rightarrow c^3 = 0 \Rightarrow c = 0$$

8. $\lim_{x\to 9} \frac{x+5}{\sqrt{x}-3} =$

[✓ (a) it does not exist]

(b) $+\infty$

(c) $-\infty$

(d) 84

(e) -84

$$\lim_{x\to 9^+} \frac{x+5}{\sqrt{x-3}} = +\infty$$

$$\lim_{x\to 9^-} \frac{x+5}{\sqrt{x}-3} = -\infty$$

9. $\lim_{x\to -\infty} \sqrt{\frac{20x^{10} - 2x^5 + 2}{5x^{10} + x^5 - 3}} =$

(a) $+\infty$

[✓ (b) 2]

(c) $-\infty$

(d) -2

(e) it does not exist

$$= \sqrt{\lim_{x\to -\infty} \frac{=}{-}}$$

$$= \sqrt{\lim_{-\infty} \frac{20x^{16}}{5x^{10}}} = \sqrt{4}$$

$$= 2$$

10. To prove that $\lim_{x\to 5}(x-2) = 3$ a reasonable relationship between δ and ϵ would be

[✓ (a)] $\delta = \epsilon$
(b) $\delta = 5\epsilon$
(c) $\delta = \sqrt{\epsilon}$
(d) $\delta = \frac{1}{\epsilon}$
(e) $\delta = \frac{1}{\sqrt{\epsilon}}$

$|f(x) - L| = |x-2-3|$

$= |x-5| < \varepsilon$

$\varepsilon = \delta$

11. Find the value of k, if possible, that will make the function continuous

$$f(x) = \begin{cases} x + 2k & x \leq 1 \\ kx^2 + x + 1 & x > 1 \end{cases}$$

(a) -1
(b) 2
[✓ (c) 1]
(d) -2
(e) none exists

$f(1) = 1+2k$

$\lim_{x\to 1^-} f(x) = 1+2k$

$\lim_{x\to 1^+} f(x) = k+2$

we need $1+2k = k+2 \Rightarrow k = 1$

12. find the limit $\lim_{x\to+\infty}\left(\cos\left(\frac{4}{x}\right)\cdot\sin\left(\frac{5}{x}\right)\right) =$

(a) 1
(b) -1
(c) $+\infty$
(d) $-\infty$
[✓ (e) 0]

$\cos\left(\lim_{x\to+\infty}\frac{4}{x}\right) \cdot \sin\left(\lim_{x\to+\infty}\frac{5}{x}\right)$

$= \cos(0) \cdot \sin(0) = 0$

13. Find an equation for the tangent line to the curve $y = x^7 - 5$ at $(1, -4)$.

- (a) $y = 7x$
- (b) $y = 7x + 5$
- (c) $y = 7x - 3$
- [✓ (d) $y = 7x - 11$]
- (e) $y = 7x - 5$

$y' = 7x^6$

slope $= y'(1) = 7$

$y - y_0 = m(x - x_0)$

$y + 4 = 7(x - 1) \Rightarrow y = 7x - 11$

14. If $y = \dfrac{2}{x+3}$, then $y'(0) =$

- [✓ (a) $\dfrac{-2}{9}$]
- (b) 0
- (c) $\dfrac{4}{9}$
- (d) $\dfrac{2}{9}$
- (e) $\dfrac{-4}{9}$

$$y' = \frac{0 - 2}{(x+3)^2} = \frac{-2}{(x+3)^2}$$

$$y'(0) = \frac{-2}{3^2} = \frac{-2}{9}$$

15. $g(x) = x^3 f(x)$. Find $g'(2)$, given that $f(2) = 6$ and $f'(2) = 3$

- (a) 48
- (b) -60
- [✓ (c) 96]
- (d) 60
- (e) -48

$g'(x) = 3x^2 f(x) + x^3 f'(x)$

$g'(2) = 3(2)^2 f(2) + 2^3 \cdot f'(2)$

$= 3(2)^2 \cdot 6 + 2^3 \cdot (3)$

$= 12 \cdot 6 + 8 \cdot 3 = 72 + 24 = 96$

16. $y = x^{-3} + x$. Find y'''

- (a) -6
- (b) $-60x^{-6} + x^{-2}$
- (c) $-60x^{-6} - x^{-2}$
- (d) $-3x^{-2} + 1$
- [✓ (e) $-60x^{-6}$]

$y' = -3x^{-4} + 1$

$y'' = 12x^{-5}$

$y''' = -60x^{-6}$

17. If $y = x^5 \cos x$, find d^2y/dx^2

(a) $20x^3 \cos x$
(b) $20x^3 \cos x - x^5 \cos x$
(c) $20x^3 \cos x + x^5 \cos x$
→ [✓ (d)] $20x^3 \cos x - 10x^4 \sin x - x^5 \cos x$
(e) $-x^5 \cos x$

$y' = 5x^4 \cos x - x^5 \sin x$

$y'' = 20x^3 \cos x - 5x^4 \sin x$
$- 5x^4 \sin x - x^5 \cos x$
$= 20x^3 \cos x - 10x^4 \sin x - x^5 \cos x$

18. Use dy to approximate $\sqrt{4.04}$ starting at $x = 4$

→ [✓ (a)] 2.01
(b) 1.99
(c) 4.01
(d) 3.99
(e) 1.59

Let $y = \sqrt{x}$ with $x_0 = 4$; $dx = 0.04$
$y_0 = \sqrt{x_0} = \sqrt{4} = 2$
$dy = \frac{1}{2\sqrt{x}} dx = \frac{1}{2\sqrt{4}} (0.04) = 0.01$
at $x = 4.04 \Rightarrow y \simeq 2 + 0.01 = 2.01$

19. Find $\frac{dy}{dx}$ if $x^2 + y^2 = 49$.

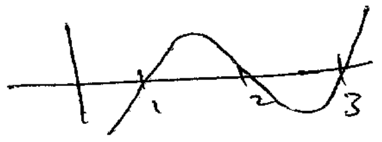
(a) $\frac{49x}{y}$
(b) $\frac{x}{y}$
→ [✓ (c)] $-\frac{x}{y}$
(d) $-\frac{49x}{y}$
(e) $\frac{y}{x}$

$x^2 + y^2 = 49$
$2x + 2yy' = 0$
$y' = -\frac{2x}{2y} \Rightarrow y' = -\frac{x}{y}$

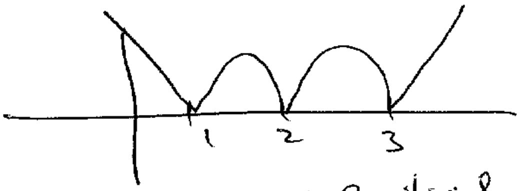
20. The number of critical points for $f(x) = |(x-1)(x-2)(x-3)|$ is

[✓ (a) 5]
(b) 2
(c) 3
(d) 4
(e) 1

graph of $y=(x-1)(x-2)(x-3)$ is



$\Rightarrow$ graph of $y=|(x-1)(x-2)(x-3)|$ is



$\Rightarrow$ # of critical points $= 5$

21. $y = \cot^{-1}\sqrt{x}$. Find dy/dx.

(a) $-\dfrac{x}{1+x}$
[✓ (b) $-\dfrac{1}{2\sqrt{x}(1+x)}$]
(c) $-\dfrac{x}{(1+x^2)}$
(d) $-\sqrt{\dfrac{1}{1+x^2}}$
(e) $\dfrac{x}{1+x}$

$$y' = -\frac{1}{1+(\sqrt{x})^2}\cdot(\sqrt{x})'$$

$$= -\frac{1}{1+x}\,\frac{1}{2\sqrt{x}} = \frac{-1}{2\sqrt{x}(1+x)}$$

22. $\displaystyle\lim_{x\to0^+}\frac{\sin x}{\ln(x^2+1)} =$

[✓ (a) $-\infty$]
(b) 10
(c) 0
[✓ (d) $+\infty$]
(e) -10

type $0/0$

$$= \lim_{x\to0^+}\frac{\cos x}{\frac{2x}{x^2+1}} = \lim_{x\to0^+}\frac{(x^2+1)\cos x}{2x}$$

$$= \lim_{x\to0^+}\frac{(x^2+1)\cos x}{2x} = +\infty$$

23. $\displaystyle\lim_{x\to0^+}(1-\ln(2x))^{2x} =$

[✓ (a) 0]
[✓ (b) 1]
(c) $+\infty$
(d) $-\infty$
(e) -1

type ∞^0

let $y = \lim_{x\to0^+}(1-\ln(2x))^{2x}$

$$\ln y = \lim_{x\to0^+}(2x)\ln[1-\ln(2x)]$$

$$= \lim_{x\to0^+}\frac{\ln[1-\ln(2x)]}{1/(2x)}$$ type ∞/∞

$$= \lim_{x\to0^+}\frac{\left(\frac{-2}{2x}\right)/[1-\ln(2x)]}{-\frac{1}{2}\frac{1}{x^2}} = \lim_{x\to0^+}\frac{\frac{1}{x}}{1-\ln(2x)}(2x^2) = \lim_{x\to0^+}\frac{2x}{1-\ln(2x)} = 0$$

24. True (T) or False (F)

(a) $f(x) = \dfrac{1}{x^3}$ on $[-1, 1]$ satisfies the hypotheses of Rolle's Theorem (F)

(b) The Mean Value Theorem can be used on $f(x) = |x - 1|$ on $[-2, 1]$ (T)

(c) $\tan x$ has a point of inflection on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ (T)

(d) $f(x) = x^{3/5}$ has a critical point (T)

(e) All relative extrema occur at critical points (T)

(f) $f(x) = |\tan^2 x|$ has no relative extrema on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ (F)

(g) $f(x) = \dfrac{x^3}{x^5 - 2}$ has no horizontal asymptote (F)

(h) $f(x) = |x^2 - 4|$ has points of discontinuity at $x = 2$ and $x = -2$. (F)

(i) The function $f(x) = \frac{x+5}{x-1}$ has a removable discontinuity at $x = 1$ (F)

(j) The slope of the tangent line to the graph of $f(x) = x^3 - 5$ at $x_0 = 3$ is 22. (F)