Formula Sheet

■■ Laplace Transforms

(a) $\mathcal{L}\{1\} = \frac{1}{s}$ (b) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ (c) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ (d) $\mathcal{L}\{sinkt\} = \frac{k}{s^2+k^2}$

(e) $\mathcal{L}\{coskt\} = \frac{s}{s^2+k^2}$ (f) $\mathcal{L}\{sinhkt\} = \frac{k}{s^2-k^2}$ (g) $\mathcal{L}\{coshkt\} = \frac{s}{s^2-k^2}$

■ Transform of a derivative: $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$

■ First Translation Theorem: $\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a} = F(s-a)$ (1)

⟹ Inverse form of (1): $\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t)$

■ Second Translation Theorem: $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$ (2)

⟹ Inverse form of (2): $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$

■ Derivatives of Transforms: $\mathcal{L}\{t^nf(t)\} = (-1)^n\frac{d^n}{ds^n}F(s)$ (3)

⟹ Inverse form of (3): $\mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\} = (-1)^nt^nf(t)$

■ Convolution Theorem: The Laplace transform of the convolution $f * g = \int_0^t f(\tau)g(t-\tau)d\tau$ is

$$\mathcal{L}\{f * g\} = F(s)G(s) \quad (4)$$

⟹ Inverse form of (4): $\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$

■ Transform of the integral of f: $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$ (5)

⟹ Inverse form of (5): $\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)d\tau$

■ Dirac delta function: $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$ (6)

⟹ Inverse form of (6): $\mathcal{L}^{-1}\{e^{-st_0}\} = \delta(t-t_0)$

■■ Fourier-Bessel Series

(i) If the $\alpha_i{}'s$ are defined by $J_n(\alpha b) = 0$, then

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad \text{where} \qquad c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x f(x) J_n(\alpha_i x) dx$$

(ii) If the $\alpha_i{}'s$ are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$, then

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \qquad \text{where} \qquad c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int_0^b x f(x) J_n(\alpha_i x) dx$$

(iii) If the $\alpha_i{}'s$ are defined by $J_0'(\alpha b) = 0$, then

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_n(\alpha_i x) \quad \text{where} \quad c_1 = \frac{2}{b^2} \int_0^b x f(x) dx, \quad c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x f(x) J_n(\alpha_i x) dx$$

■■ Fourier-Legendre Series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where
$$c_n = \tfrac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

and the Legendre polynomials can be generated by using the following Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n = 0,1,2,\ldots$$

■■ Operational properties of Fourier sine and cosine transforms

(A) $\mathcal{F}_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0)$

(B) $\mathcal{F}_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0)$