

1. (4-points) Use the comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5n^2 + 2n}{(1+n^2)^2}$ converges or diverges.

$$a_n = \frac{5n^2 + 2n}{(1+n^2)^2} < \frac{5n^2 + 2n}{n^4} = \frac{5}{n^2} + \frac{2}{n^3}$$

$$\text{But } \sum_{n=1}^{\infty} \frac{5}{n^2} = 5 \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{2}{n^3} = 2 \sum_{n=1}^{\infty} \frac{1}{n^3}$$

are both convergent p -series ($p=2$ and $p=3$).

Therefore, the given series is convergent by the comparison test.

2. (4-points) Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{1-n+5n^2}{\sqrt{1+23n^2+4n^6}}$ converges or diverges.

$$\text{Let } a_n = \frac{1-n+5n^2}{\sqrt{1+23n^2+4n^6}} \quad \text{and} \quad b_n = \frac{1}{n} \Rightarrow$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1-n+5n^2}{\sqrt{1+23n^2+4n^6}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} - \frac{1}{n} + 5}{\sqrt{\frac{1}{n^6} + \frac{23}{n^4} + 4}} \\ &= \frac{5}{\sqrt{4}} = \frac{5}{2} \text{ which is finite and } \neq 0 \end{aligned}$$

Therefore, the given series is divergent by the limiting comparison test since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series.

3. (4-points) Use the alternating series test to determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-\frac{n}{5}}$ converges or divergence.

$$b_n = n e^{-\frac{n}{5}} > 0. \text{ Let } f(x) = x e^{-\frac{x}{5}} \Rightarrow$$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x}{e^{\frac{x}{5}}} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{5} e^{\frac{x}{5}}} = 0$$

$$\text{and } f'(x) = e^{-\frac{x}{5}} - \frac{1}{5} x e^{-\frac{x}{5}} = \frac{1}{5} e^{-\frac{x}{5}} (5 - x)$$

$\Rightarrow f'(x) < 0$ for $x > 5 \Rightarrow \{b_n\}$ is eventually decreasing.

Therefore, the series is convergent by the alternating series test.

4. (4-points) Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{(4n+1)^5}{3^n}$ for absolute convergence.

$$a_n = (-1)^n \frac{(4n+1)^5}{3^n} \Rightarrow a_{n+1} = (-1)^{n+1} \frac{(4n+5)^5}{3^{n+1}}$$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(4n+5)^5}{3^{n+1}} \cdot \frac{3^n}{(4n+1)^5} \right| = \frac{1}{3} \left(\frac{4 + \frac{5}{n}}{4 + \frac{1}{n}} \right)^5$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{3} < 1 \Rightarrow \text{The series}$$

is absolutely convergent according to the ratio test.