King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Math 101- Calculus I
Exam I
2007-2008 (073)

Tuesday, July 22, 2008 **Allowed Time: 2 hours**

Name: Solution Key

ID Number: ____________

Section Number: ____________ **Serial Number:** ____________

Instructions:

1. Write neatly and eligibly. You may lose points for messy work.
2. **Show all your work.** No points for answers without justification.
3. **Calculators and Mobiles are not allowed.**
4. Make sure that you have 8 different problems (6 pages + cover page)

Problem No	Grade	Maximum Points
1		33
2		7
3		7
4		8
5		8
6		13
7		8
8		16
Total		100

MATH 101, EXAM-I (Term 073)

1. Evaluate the limit if it exists. Justify your answer

(a) $\lim\limits_{x\to 0^+} \dfrac{x-1}{x^2+2x}$. (4 pts.)

2 As $x \to 0^+$, $x-1 \to -1$ & x^2+2x goes to zero from the right (through positive values).

2 Thus $\lim\limits_{x\to 0^+} \dfrac{x-1}{x^2+2x} = -\infty$

(b) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$. $\frac{0}{0}$, undefined. (8 pts.)

Multiply by the Conjugates:

$$\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} \cdot \frac{\sqrt{x}+x^2}{\sqrt{x}+x^2} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \qquad 1.5+1.5$$

$$= \lim_{x\to 1} \frac{x-x^4}{1-x} \cdot \frac{1+\sqrt{x}}{\sqrt{x}+x^2} \qquad 1$$

$$= \lim_{x\to 1} \frac{x(1-x)(1+x+x^2)}{1-x} \cdot \frac{1+\sqrt{x}}{\sqrt{x}+x^2} \qquad 1$$

$$= \lim_{x\to 1} x(1+x+x^2) \cdot \frac{1+\sqrt{x}}{\sqrt{x}+x^2} \qquad 1$$

$$= 1 \cdot 3 \cdot \frac{2}{2} = 3 \qquad 2$$

(c) $\lim\limits_{x\to 0^-} x\sin\left(\dfrac{\sqrt{x+2}}{x}\right)$. (6 pts.)

$$-1 \le \sin\left(\frac{\sqrt{x+2}}{x}\right) \le 1 \qquad 1$$

$$\Rightarrow \quad -x \ge x\sin\left(\frac{\sqrt{x+2}}{x}\right) \ge x. \quad (x<0 \text{ since } x \to 0^-) \qquad 1$$

Since $\lim\limits_{x\to 0^-} -x = 0$ & $\lim\limits_{x\to 0^-} x = 0$, 1 + 1

then, by the Squeeze Theorem, 1

$$\lim x\sin\left(\frac{\sqrt{x+2}}{x}\right) = 0 \qquad 1$$

(d) $\lim_{x\to 1} \arcsin\left(\frac{1-x}{1-x^2}\right)$. Since arcsin is Continuous, then (4 pts.)

$$\lim_{x\to 1} \arcsin\left(\frac{1-x}{1-x^2}\right) = \arcsin\left(\lim_{x\to 1} \frac{1-x}{(1-x)(1+x)}\right) \quad (1)$$

$$= \arcsin\left(\lim_{x\to 1} \frac{1}{1+x}\right) \quad (1)$$

$$= \arcsin\left(\frac{1}{2}\right) \quad (1)$$

$$= \frac{\pi}{6} \quad (1)$$

(e) $\lim_{x\to -\infty} \frac{x^3-2x+7}{-2x^2+x-3}$. (4 pts.)

Divide the numerator & Denominator by x^2:

$$\lim_{x\to -\infty} \frac{x^3-2x+7}{-2x^2+x-3} = \lim_{x\to -\infty} \frac{x - \frac{2}{x} + \frac{7}{x^2}}{-2 + \frac{1}{x} - \frac{3}{x^2}} \quad (2)$$

$$= \frac{-\infty - 0 + 0}{-2 + 0 - 0} \quad (1)$$

$$= +\infty \quad (1)$$

(f) $\lim_{x\to +\infty} \left(\sqrt{9x^2+x}-3x\right)$. $\infty - \infty$, undefined (7 pts.)

$$\lim_{x\to +\infty} \sqrt{9x^2+x} - 3x \cdot \frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} \quad (1)$$

$$= \lim_{x\to +\infty} \frac{x}{\sqrt{9x^2+x}+3x} \quad (1)$$

$$= \lim_{x\to +\infty} \frac{x}{|x|\sqrt{9+\frac{1}{x}}+3x} \quad (1)$$

$$= \lim_{x\to +\infty} \frac{x}{x\sqrt{9+\frac{1}{x}}+3x} \quad (2) \quad (|x| = x \text{ since } x \to +\infty)$$

$$= \lim_{x\to +\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} \quad (1)$$

$$= \frac{1}{\sqrt{9+0}+3} = \frac{1}{6} \quad (1)$$

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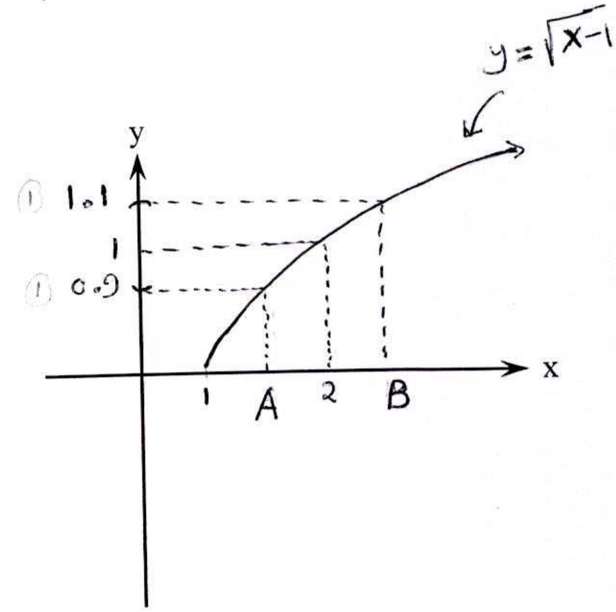
2. Use the graph of $f(x)=\sqrt{x-1}$ to find a number δ such that (7 pts.)

$$\left|\sqrt{x-1}-1\right|<0.1 \text{ whenever } |x-2|<\delta.$$

- $\sqrt{x-1}=0.9 \Rightarrow x-1=0.81$
  $\Rightarrow x=1.81=A$ (1)
- $\sqrt{x-1}=1.1 \Rightarrow x-1=1.21$
  $\Rightarrow x=2.21=B$ (1)
- $\delta=\text{minimum}(2-A, B-2)$ (2)
  $=\text{minimum}(0.19, 0.21)$
  $=0.19$ (1)

(or any smaller positive number)



3. Where is the function $f(x)=\dfrac{1}{1-e^{\frac{x+1}{x}}}$ continuous? (7 pts.)

f is continuous in its domain

(3) - Because of the fraction $\frac{x+1}{x}$, we must have $x\neq 0$.

(3) - We also must have the denominator $1-e^{\frac{x+1}{x}}\neq 0$. Thus

$\frac{x+1}{x}\neq 0 \Rightarrow x\neq -1$.

(1) - So f is continuous everywhere except at $x=0$ & $x=-1$

or f is continuous in $(-\infty,-1)\cup(-1,0)\cup(0,+\infty)$

MATH 101, EXAM-I (Term 073)

4. Find the constant k that makes the function (8 pts.)

$$f(x)=\begin{cases} x^2-k^2 & \text{if } x\le 2 \\ kx+5 & \text{if } x>2 \end{cases}$$

continuous on $(-\infty,+\infty)$.

- If $x<2$, then $f(x)=x^2-k^2$ is continuous for all values of k (a polynomial)

(1) • If $x>2$, then $f(x)=kx+5$ " " " " " "

- We need to check continuity at $x=2$. We need to find k so that

$$\lim_{x\to 2} f(x) = f(2):$$

$$* \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2-k^2 = 4-k^2$$

(3)

$$* \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} kx+5 = 2k+5$$

(2) For the limit to exist, we must have $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$; i.e,

(2) $4-k^2 = 2k+5 \Rightarrow k^2+2k+1=0 \Rightarrow (k+1)^2=0 \Rightarrow \underline{\underline{k=-1}}.$

With this choice of k, f will be conts at $x=2$.

Thus the value of k that will make f continuous on $(-\infty,+\infty)$ is $k=-1$.

5. Show that the equation $x\ln x=\sin x$ has a root in the interval $(1,e)$. (8 pts.)

Apply the Intermediate Value Theorem by letting

(2) $f(x)=x\ln x-\sin x$, $[1,e]$, $N=0$

(2) • f is continuous on $[1,e]$ since $x\ln x$ & $\sin x$ are continuous on $[1,e]$

(1) • $f(1)=1\cdot\ln 1-\sin 1=-\sin 1<0$

(1) $f(e)=e\cdot\ln e-\sin e = e-\sin e>0$

So $N=0$ is between $f(1)$ & $f(e)$

By the Intermediate Value Theorem, there is a number c in $(1,e)$ such that

(2)

$$f(c)=0,$$

that is, $c\ln c-\sin c=0$

or $c\ln c=\sin c.$

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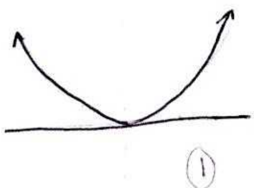
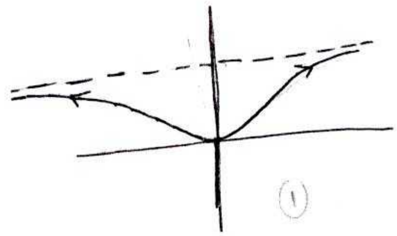
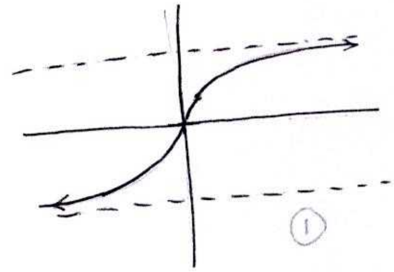
6. (a) How many horizontal asymptotes can a function have? (6 pts.)
Illustrate your answer graphically.

There are 3 possibilities.

. None (1) . One (1) . Two (1)







(b) Does the graph of $f(x) = \ln\left(9 - x^2\right)$ have a vertical asymptote (7 pts.)

(i) at $x = 3$. Justify.

Since $\lim_{x \to 3^-} \ln(9 - x^2) = -\infty$ (3), then f has a vertical asymptote at $x = 3$ (1)

(ii) at $x = -1$. Justify.

Since $\lim_{x \to -1} \ln(9 - x^2) = \ln 8 \neq \pm\infty$ (2), then f has no vertical asymptote at $x = -1$. (1)

7. The position function of a particle moving in a straight line is given by the equation of motion $s(t) = \dfrac{1-t}{1+t}$, where t is measured in seconds and s in meters. Find the instantaneous velocity of the particle when $t = 1$. (8 pts.)

$$v(1) = \lim_{h \to 0} \frac{s(1+h) - s(1)}{h} \quad (3)$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1-(1+h)}{1+(1+h)} - 0\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-h}{2+h}\right] \quad (3)$$

$$= \lim_{h \to 0} \frac{-1}{2+h}$$

$$= -\frac{1}{2} \ \text{m/s} \quad (2)$$

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8. (a) **TRUE** or **FALSE**. Justify: If $f'(a)$ exists, then $\lim_{x\to a} f(x) = f(a)$. (4 pts.)

TRUE as $f'(a)$ exists $\Rightarrow f$ is differentiable at a 1
1
$\Rightarrow f$ is continuous at a 1
$\Rightarrow \lim_{x\to a} f(x) = f(a)$ 1

(b) Is $f(x) = x|x|$ differentiable at $x = 0$. Justify. (6 pts.)

We Calculate $f'(0)$.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \quad 2$$

$$= \lim_{x\to 0} \frac{x|x| - 0}{x - 0}$$

$$= \lim_{x\to 0} \frac{x|x|}{x} = \lim_{x\to 0} |x| = 0. \quad 2$$

Since $f'(0)$ exists, then f is differentiable at $x=0$. 2

(c) Graph the derivative of the function whose graph is given below. (6 pts.)