MATH 260-1-3 (072) EXAM I (KEY)

1. (5-points) Find all values of the real number m for which $y = x^m$ is a solution of the DE $x^2y'' - 5xy' + 8y = 0$.

$y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$ $\Rightarrow$

$x^2y'' - 5xy' + 8y = m(m-1)x^m - 5mx^m + 8x^m = 0$

$\Rightarrow x^m[m^2 - 6m + 8] = 0 \Rightarrow m^2 - 6m + 8 = 0$

$\Rightarrow (m-2)(m-4) = 0 \Rightarrow$ The possible values of m are 2 and 4 #

2. (5-points) Find a DE of the form $\frac{dy}{dx} = f(x, y)$ which represents a family of curves with the property that every line normal to a graph of a member of the family passes through the point $(-1, 2)$.

The graph of any member of the family must be a circle with center at $(-1, 2)$ $\Rightarrow$

$(x+1)^2 + (y-2)^2 = r^2$, r is any real number

$\Rightarrow 2(x+1) + 2(y-2)y' = 0 \Rightarrow$

$$\frac{dy}{dx} = \frac{x+1}{2-y}.$$ #

6. (6-points) Solve $x\dfrac{dy}{dx} = 3y + x^4 \cos x$.

It is a linear first-order DE, we put in standard form $\Longrightarrow$

$$\frac{dy}{dx} - \frac{3}{x}y = x^3 \cos x$$

$$\text{I.F} = e^{-\int \frac{3}{x}dx} = e^{-3\ln x} = e^{\ln \frac{1}{x^3}} = \frac{1}{x^3}$$

The G.S: $\dfrac{1}{x^3}y = \int \left(\dfrac{1}{x^3}\right)(x^3 \cos x)\, dx + C$

$\Longrightarrow \quad y = x^3 [\sin x + C]$. #.

7. (5-points) Show that the DE

$$3(1+y^2)dx - 2xy(x^3-1)dy = 0$$

is of Bernoulli's type. Then use a suitable substitution which transforms the DE into a linear first-order DE. [Find the new DE but DO NOT solve it].

Look at $\dfrac{dx}{dy}$ instead of $\dfrac{dy}{dx}$ $\Longrightarrow$

$$3(1+y^2)\frac{dx}{dy} + 2yx = 2yx^4 \Longrightarrow$$

$$3x^{-4}\frac{dx}{dy} + \frac{2y}{1+y^2}x^{-3} = \frac{2y}{1+y^2} \quad \text{(i)}$$

which is of Bernoulli's type.

Put $u = x^{-3} \Longrightarrow \dfrac{du}{dx} = -3x^{-4}\dfrac{dx}{dy}$ (ii)

(i), (ii) $\Longrightarrow \quad -\dfrac{du}{dx} + \dfrac{2y}{1+y^2}u = \dfrac{2y}{1+y^2}$ #

8. (5-points) Determine the values of α and β for which the system

$$x - 3y = 1$$
$$2x + \alpha y = \beta$$

has (a) a unique solution; (b) no solution; (c) infinitely many solutions.

$$\Longrightarrow \left[\begin{array}{cc|c} 1 & -3 & 1 \\ \boxed{2} & \alpha & \beta \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 6+\alpha & -2+\beta \end{array}\right]$$

(a) Unique solution if $\alpha \neq -6$ and β is any real no.

(b) no solution if $\alpha = -6$ and $\beta \neq 2$

(c) Infinitely many solutions if $\alpha = -6$ and $\beta = 2$

#

9. (5-points) Find the reduced row-echelon form of the augmented matrix of the homogeneous system:

$$3x_1 - x_2 + 2x_3 - 4x_4 = 0$$
$$x_1 + 3x_2 - 6x_3 + 2x_4 = 0$$
$$7x_1 + 11x_2 - 22x_3 + 4x_4 = 0$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 3 & -1 & 2 & -4 & 0 \\ 1 & 3 & -6 & 2 & 0 \\ 7 & 11 & -22 & 4 & 0 \end{array}\right]$$

then find the solution set of the system.

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 3 & -6 & 2 & 0 \\ \boxed{3} & -1 & 2 & -4 & 0 \\ \boxed{7} & 11 & -22 & 4 & 0 \end{array}\right] \xrightarrow[-7R_1+R_3]{-3R_1+R_2} \left[\begin{array}{cccc|c} 1 & 3 & -6 & 2 & 0 \\ 0 & -10 & 20 & -10 & 0 \\ 0 & \boxed{-10} & 20 & -10 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3} \left[\begin{array}{cccc|c} 1 & 3 & -6 & 2 & 0 \\ 0 & -10 & 20 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-\frac{1}{10}R_2} \left[\begin{array}{cccc|c} 1 & \boxed{3} & -6 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$ which is now in reduced row echelon form

let $x_3 = \alpha$ and $x_4 = \beta \Longrightarrow$ any solution is of the form $(\beta, 2\alpha - \beta, \alpha, \beta)$ where α, β are any real nos.

Bonus Problem (Choose one only)

10. **Bonus Problem 1 (5-points)** The population of bacteria in culture decays at a rate proportional to the number of bacteria present at time t. After 12 hours it is observed that there are 10000 bacteria present. After 21 hours there are only 500 bacteria present. Find the DE that models this case.
[You may use the approximation: $\ln 20 \approx 3$, $e^4 \approx 55$, $e^7 \approx 1100$]

$$\frac{dP}{dt} \propto P \Longrightarrow \frac{dP}{dt} = kP \Longrightarrow$$

$$\boxed{P = ce^{kt}}$$

Need to find c and k

$$t = 12 \Rightarrow P = 10000 \Longrightarrow 10000 = ce^{12k} \quad (i)$$

$$t = 21 \Rightarrow P = 500 \Longrightarrow 500 = ce^{21k} \quad (ii)$$

$$(i) \div (ii) \Longrightarrow 20 = e^{-9k} \Rightarrow -9k = \ln 20$$

$$\Longrightarrow k = -\frac{1}{9}\ln 20 \approx \left(-\frac{1}{9}\right)(3) = -\frac{1}{3}$$

$$\Longrightarrow \boxed{k = -\frac{1}{3}}$$. Substitute in (i) $\Longrightarrow$

$$10000 = ce^{-4} \Longrightarrow c = (10000)e^{4}$$

$$\Longrightarrow c \approx (10000)(55) = 550000 \Longrightarrow$$

The required DE is

$$\frac{dP}{dt} = -\frac{1}{3}P \quad \#$$

11. **Bonus Problem 2 (5-points)** Solve the DE

$$2x \tan y \sec^2 y \frac{dy}{dx} = 4x^2 + \tan^2 y. \quad (i)$$

Put $u = \tan^2 y \Rightarrow \frac{du}{dx} = 2 \tan y \sec^2 y \frac{dy}{dx}$ (ii)

(i) & (ii) $\Rightarrow$

$$x \frac{du}{dx} = 4x^2 + u$$

$$\Rightarrow \quad \frac{du}{dx} - \frac{1}{x} u = 4x.$$

which is a linear DE of the first order

$$I.F = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x} \Rightarrow$$

The G.S. is

$$\frac{1}{x} u = \int \left(\frac{1}{x}\right)(4x)\, dx + C \Rightarrow$$

$$\tan^2 y = x \left[4x + C \right] \Rightarrow$$

$$\boxed{\tan^2 y = 4x^2 + Cx}$$ #