KFUPM SEM II (Term 072) Name: ____________

MATH 260-1-3 Quiz# 4 ID #: KEY Sec.#: __________ Serial #: __________

1. (7-points) Find a nonsingular matrix P such that $P^{-1}\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} P$ is a diagonal matrix.

First we find the eigenvalues of the matrix $\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = -4-3\lambda+\lambda^2+6 = \lambda^2-3\lambda+2 = (\lambda-1)(\lambda-2)=0.$$

$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2$

$\boxed{\text{For } \lambda_1 = 1}$ solve $\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = 1$

$\boxed{\text{For } \lambda_2 = 2}$ solve $\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 2 \Rightarrow$ The required matrix

$$P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

one checks that $P^{-1}\begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$

2. (4-points) Given that $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$ are the eigenvalues of the matrix $\begin{bmatrix} -2 & 4 & -1 \\ -3 & 5 & -1 \\ -1 & 1 & 1 \end{bmatrix}$. Determine whether or not A is diagonalizable.

$\boxed{\text{For } \lambda_1 = \lambda_2 = 1}$ solve $\left[\begin{array}{ccc|c} -2-\lambda & 4 & -1 & 0 \\ -3 & 5-\lambda & -1 & 0 \\ -1 & 1 & 1-\lambda & 0 \end{array}\right] \Rightarrow$

$\left[\begin{array}{ccc|c} -3 & 4 & -1 & 0 \\ \boxed{-3} & 4 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ \boxed{-3} & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 = x_2 = x_3 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace E_1 associated with the eigenvalue 1 of multiplicity 2 $\Rightarrow \dim E_1 = 1 \Rightarrow$ A is not diagonalizable (To be diagonalizable $\dim E_1$ must be 2).

3. (4-points) Use Cayley-Hamilton Theorem to find A^3, where $A = \begin{bmatrix} 0 & 0 & 0 \\ 250 & 0 & 2 \\ -100 & 2 & 0 \end{bmatrix}$.

(No other method of finding A^3 will be accepted).

According to Cayley-Hamilton Theorem, A satisfies its characteristic equation $\begin{vmatrix} -\lambda & 0 & 0 \\ 250 & -\lambda & 2 \\ -100 & 2 & -\lambda \end{vmatrix}$

$= -\lambda(\lambda^2 - 4) = 0 \Rightarrow \lambda^3 - 4\lambda = 0 \Rightarrow$

$A^3 - 4A = 0 \Rightarrow A^3 = 4A = \begin{bmatrix} 0 & 0 & 0 \\ 1000 & 0 & 8 \\ -400 & 8 & 0 \end{bmatrix}$.

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1. (7-points) Find a nonsingular matrix P such that $P^{-1}\begin{bmatrix} 10 & 8 \\ -7 & -5 \end{bmatrix} P$ is a diagonal matrix.

First we find the eigenvalues of the matrix $\begin{bmatrix} 10 & 8 \\ -7 & -5 \end{bmatrix}$

$$\begin{vmatrix} 10-\lambda & 8 \\ -7 & -5-\lambda \end{vmatrix} = -50 - 5\lambda + \lambda^2 + 56 = \lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow (\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$$

For $\lambda_1 = 2$ | solve $\left[\begin{array}{cc|c} 8 & 8 & 0 \\ -7 & -7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_1 = 2$

For $\lambda_2 = 3$ | solve $\left[\begin{array}{cc|c} 7 & 8 & 0 \\ -7 & -8 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 7 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

$v_2 = \begin{bmatrix} 8 \\ -7 \end{bmatrix}$ is an eigenvector associated with the eigenvalue $\lambda_2 = 3$ $\Rightarrow$ The required matrix P is $P = \begin{bmatrix} 1 & 8 \\ -1 & -7 \end{bmatrix}$.

one checks that $P^{-1}\begin{bmatrix} 10 & 8 \\ -7 & -5 \end{bmatrix} P = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

• 2. (4-points) Given that $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 1$ are the eigenvalues of the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$. Determine whether or not A is diagonalizable.

$\boxed{\text{For } \lambda_1 = \lambda_2 = 2}$ solve $\left[\begin{array}{ccc|c} 3-\lambda & -2 & 1 & 0 \\ 1 & -\lambda & 1 & 0 \\ -1 & 1 & 2-\lambda & 0 \end{array}\right] \Longrightarrow$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \boxed{-2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow x_1 = x_2 = x_3 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for the eigenspace E_2 associated with the eigenvalue 2 of multiplicity 2 $\Rightarrow \dim E_2 = 1 \Longrightarrow$

A is not diagonalizable (To be diagonalizable, $\dim E_2$ must equal to 2).

3. (4-points) Use Cayley-Hamilton Theorem to find A^3, where $A = \begin{bmatrix} 0 & 0 & 0 \\ 100 & 0 & 3 \\ -300 & 3 & 0 \end{bmatrix}$.

(No other method of finding A^3 will be accepted).

According to Cayley-Hamilton Theorem, A satisfies its characteristic equation $\begin{vmatrix} -\lambda & 0 & 0 \\ 100 & -\lambda & 3 \\ -300 & 3 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 9) = 0$

$\Rightarrow \lambda^3 - 9\lambda = 0 \Longrightarrow A^3 - 9A = 0 \Longrightarrow$

$$A^3 = 9A = \begin{bmatrix} 0 & 0 & 0 \\ 900 & 0 & 27 \\ -2700 & 27 & 0 \end{bmatrix}.$$