King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

**Final Report**

Research Project Number FT/2002-01

# DETERMINISTIC AND RANDOM VERSIONS OF FAN'S APPROXIMATION THEOREM WITH APPLICATIONS

By

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# Final Report

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**Abstract (English)**

The goal of this project is to study approximation results for multivalued continuous and *-nonexpansive maps on both compact convex and noncompact convex subsets of metrizable topological vector spaces and hyperconvex spaces. Our main tool will be the well known Ky Fan's intersection lemma. We will mainly focus on deterministic and random versions of Fan's approximation theorem for multivalued continuous and *-nonexpansive maps on a metrizable topological vector space. As applications of our results we aniticpate that some well known theorems in approximation theory would follow as corollaries to our results, thus broaderning the scope of approximation theory.

**Key Words and Phrases**: Best approximation, fixed point, multivalued random operator, quasi-convex function, metrizable topological vector space, Kirzbraun property, random fixed point, random approximation, nonexpansive map, Banach operator, contractive family, *-nonexpansive multivalued map, hyperconvex space, reducing space, projection.

# Contents

# Introduction

Let $C$ be a nonempty subset of a normed space $X$ and $f : C \to X$, a map. A solution to the functional equation $fx = x$ will be an element $u$ in $C$ such that $fu = u$. In the case of nonexistence of a solution to the equation $fx = x$, it is natural to explore the existence of an optimal approximate solution that will fulfill the requirement to some extent. In other words, an element $u$ in $C$ should be found so that

$$\|u - fu\| = d(fu, C) = \inf\{d(fu, x) : x \in C\}. \tag{1}$$

This leads to finding a solution to the optimization problem $\min\{d(x, fx) : x \in C\}$. Note that $y$ is a solution of (1) if and only if $y$ is a fixed point of $P \circ f$, where $P$ is the metric projection of $X$ onto $C$. If $f$ satisfies a suitable boundary condition, then the set of solutions of (1) coincides with the fixed point set of $f$ (see Park [37]). This and some other situations explain a close relationship between fixed points and best approximations.

Approximation theory has applications in analysis, artifical neural networks, wavelets and engineering. Fixed point theorems for multifunctions are useful for many problems in control theory, game theory, optimization and economics (see for instance part I of [52]); in particular these theorems have been used extensively in approximation theory (see for example [40]). Random operators lie at the heart of probabilistic functional analysis and their theory is needed for the study of various classes of random equations. It is also worth mentioning that as applications of random fixed point theorems, a number of existence theorems for random approximation theory, random nonlinear Hammerstein equations and stochastic partial differential equations have been given by many authors. The study of multivalued fixed points have gained tremendous importance with the work of Agarwal and O'Regan [1], Beg and Shahzad [7], Engl [15], Papageorgiou [35], Sehgal and Singh [40], and Xu [50].

For all practical purposes, approximation theory permits optimal modelling in all numerical methods for approximating processes; for specific applications of the approximation theory as a branch of optimization, we refer to 37.12 and 37.13 in part III of [52] which deal with deterministic and stochastic compensation analysis and control problems.

The concept of a $^*$-nonexpansive multivalued map was introduced and studied by Husain and Latif [21] which is a generalization of the usual notion of nonexpansiveness for single-valued maps. In general, $^*$-nonexpansive multivalued maps are neither nonexpansive nor continuous (see Example 2.2.1).

Xu [50] has established some fixed point theorems while inter-play between best approximation and fixed point results for $^*$-nonexpansive maps in the context of Banach spaces and Fréchet spaces has been studied in [25-28].

This project is concerned with a study of approximation theory in metrizable topological vector spaces and hyperconvex spaces with particular reference to the Ky Fan's best approximation theorem. We also focus on nonexpansive multivalued maps, *-nonexpansive multivalued maps, affine maps, quasiconvex maps and proximity maps. Banach spaces also play an important role in the theory of best approximations and related fields. In this project, we have studied some fundamental properties of Banach operators in the context of decomposition properties and a functional equation. The relationship between decompositions and approximation results still remains an open problem.

This report is organized as follows. In Chapter 1, we prove some results about random fixed point theorems and random approximations which are stochastic generalizations of classical fixed point and approximation theorems. We obtain random fixed point generalizations of certain fixed point theorems of Dotson [14] (see also [19], [49, Theorem 1]). As an application, we prove Brosowski-Meinardus theorem on invariant approximation (see [9, 28, 33]).

Among other results, we show that if $S$ is a compat subset of a Banach space $X$, $F$ a family of contractive jointly continuous functions associated with $S$, $T : \Omega \times S \to S$, ($\Omega = [0, 1]$), a nonexpansive random operator, then $T$ has a random fixed point. The contents of this chapter form a paper: [A.R. Khan, A.B. Thaheem and N. Hussain, Random fixed points and random approximations, Southeast Asian Bull. Math. 27(2003), 289–294].

In Chapter 2, we establish some new deterministic forms (versions) of Ky Fan's best approximations. We also establish some new deterministic forms of approximation results for continuous maps and a discontinuous class of multivalued maps (*-nonexpansive maps) on compact convex and noncompact convex sets in metrizable topological vector spaces and hyperconvex spaces. In fact, our results present mutlivalued analog of some well known approximation theorems for hyperconvex spaces. The contents of this chapter form a paper: [A.R. Khan, N. Hussain and A.B. Thaheem, Some generalizations of Ky Fan's best approximation theorem, Analysis in Theory and Applications 20(2004), 189–198].

Chapter 3 deals with a stochastic version of Fan's best approximation theorem. In Section 3.3, we prove some approximation results for single-valued continuous quasi-convex mappings on compact as well as on noncompact subsets of a metrizable topological vector space.

In Section 3.4, we obtain random versions of some results from Section 3.3. These results in turn extend Theorem 4 of [8] and Theorem 5 of [41] to the general framework of metrizable topological vector spaces. The contents of this chapter form a paper: [A.R. Khan, A.B. Thaheem and N. Hussain, A stochastic version of Fan's best approximation theorem, J. Appl. Stoch. Anal. 16(2003), 275–282.

In Chapter 4, we study some properties of Banach operators which generalize contractions and play an important role in the fixed point theory; their consideration goes back to Cheney and Goldstein [13] in the study of proximity maps on convex sets (see [34] and references therein). We study some decomposition results related to Banach operators. The relationship between decompositions and the associated approximation theory needs to be explored. The contents of this chapter form a paper: [A.B. Thaheem and A.R. Khan, On some properties of Banach operators, II, Internat. J. Math. Math. Sci. 47(2004), 2513–2515.

We have included preliminary material in all the chapters for reader's convenience.

# Chapter 1

# RANDOM FIXED POINTS AND RANDOM APPROXIMATIONS

## 1.1 Introduction and Preliminaries

Random fixed point theorems and random approximations are stochastic generalizations of classical fixed point and approximation theorems, and have applications in probability theory and nonlinear analysis. The random fixed point theory for self-maps and nonself-maps has been developed during the last two decades by various authors (see e.g. [4-5, 20, 32, 35]). Recently, this theory has been further extended for 1-set-contractive mappings that include condensing, nonexpansive, semicontractive and completely continuous random maps, etc. The aim of this chapter is to give random fixed point generalizations of certain fixed point theorems obtained by Dotson [14] (see also [19] and [49, Theorem 1]). As applications of our results, we prove Brosowski-Meinardus theorem on invariant approximation (see [9, 33]).

We recall some preliminary notions and fix our terminology for the development of our results. Let $\mathcal{A}$ be the Lebesgue $\sigma$-algebra of subsets of $\Omega = [0,1]$ and $X$ a Banach space. A mapping $T : \Omega \to 2^X$ is said to be measurable if for any open subset $C$ of $X$, $T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in \mathcal{A}$. Let $S$ be a nonempty subset of $X$. A mapping $f : \Omega \times S \to X$ is called a random operator if for any $x \in S$, $f\ (\cdot, x)$ is measurable. A measurable mapping $\xi : \Omega \to S$ is called a random fixed point of a random operator $f : \Omega \times S \to X$ if for every $\omega \in \Omega$, $\xi(\omega) = f(\omega, \xi(\omega))$. A map $f : S \to X$ is called nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in S$. Following Naimpally, Singh and Whitfield [34], $f$ is called a Banach operator if there is a constant $k$, $\ 0 \leq k < 1$ such that $\|f(x) - f^2(x)\| \leq k\|x - f(x)\|$, for all $x \in S$. A random operator $f : \Omega \times S \to X$ is continuous (nonexpansive, Banach operator, etc.) if for each $\omega \in \Omega$, $f(\omega, \cdot)$ is continuous (nonexpansive Banach operator, etc.). Let $F = \{f_x\}_{x \in S}$ be a family of functions from $[0,1]$ into $S$ with the property that $f_x(1) = x$. Following Dotson [14] we shall say that the family $F$ is contractive if there is a function $\phi : (0,1) \to (0,1)$ such that $\|f_x(t) - f_y(t)\| \leq \phi(t)\|x - y\|$ for all $x, y \in S$ and $t \in [0,1]$. The family $F$ is said to be jointly continuous if $t \to t_0$ in $[0,1]$ and $x \to x_0$ imply $f_x(t) \to f_{x_0}(t_0)$ in $S$ and $F$ is said to be jointly weakly continuous if $t \to t_0$ in $[0,1]$ and $x \xrightarrow{\omega} x_0$ imply $f_x(t) \xrightarrow{\omega} f_{x_0}(t_0)$ in $S$ (convergence here is the weak convergence). For any $x \in X$, we set $d(x, S) = \inf\{\|x - y\| : y \in S\}$ and $P_S(x) = \{y \in S : \|x - y\| = d(x, S)\}$; $P_S(x)$ is called the set of all best approximations of $x$ from $S$ in $X$.

## 1.2 Main Results

Random approximation theory and random fixed point results have received much attention since the publication of survey paper by Bharucha-Reid [10] in 1976. In this section we prove some random fixed point and random approximation theorems that generalize the corresponding fixed point and approximation theorems (see e.g. [14, 19, 25]).

**Theorem 1.2.1** *Let $S$ be a compact subset of a Banach space $X$, $F$ a family of contractive jointly continuous functions associated with $S$ and $T : \Omega \times S \to S$ a nonexpansive random operator. Then $T$ has a random fixed point.*

**Proof.** Let $\lambda_n = \frac{n}{n+1}$. Define $T_n : \Omega \times S \to S$ by $T_n(\omega, x) = f_{T(\omega,x)}(\lambda_n), \quad n = 1, 2, \ldots$ Then $T_n$ maps $S$ into $S$ and each $T_n$ is continuous because of the joint continuity of $f_x(t) \quad (x \in S, \quad t \in [0,1])$. We first show that each $T_n$ is a Banach operator.

$$\begin{aligned}
\|T_n(\omega,x) - T_n^2(\omega,x)\| &= \left\| f_{T(\omega,x)}(\lambda_n) - f_{T\left(\omega, f_{T(\omega,x)}(\lambda_n)\right)}(\lambda_n) \right\| \\
&\leq \phi(\lambda_n) \left\| T(\omega,x) - T\Big(\omega, f_{T(\omega,x)}(\lambda_n)\Big) \right\| \\
&\leq \phi(\lambda_n) \left\| x - f_{T(\omega,x)}(\lambda_n) \right\| \\
&= \phi(\lambda_n) \left\| x - T_n(\omega,x)\Big) \right\|
\end{aligned}$$

for each $\omega \in \Omega$. By continuity of $T_n(\cdot, x), \quad (x \in S)$, the inverse image of any open subset $C$ of $S$ is open in $\Omega = [0,1]$ and hence Lebsegue measurable. Thus each $T_n(\cdot, x)$ is a random operator. By [4, Theorem 2.1], $T_n$ has a random fixed point $\xi_n$. For each $n$, define $G_n : \Omega \to K(S)$ by $G_n(\omega) = \overline{\{\xi_i(\omega) : \ i \geq n\}}$

where $K(S)$ is the set of all nonempty compact subsets of $S$. Define $G : \Omega \to K(S)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Then $G$ is measurable (see [5]) and hence it has a measurable selector $\xi$. We show that $\xi$ is the random fixed point of $T$. Fix $\omega \in \Omega$. The compactness of $\overline{T(\omega, S)}$ implies that $(T(\omega, \xi_n(\omega))$ has a subsequence $\big(T(\omega, \xi_{n_j}(\omega)\big)$ which converges to $\xi(\omega)$. As $\lambda_{n_j} \to 1$, the joint continuity implies $\xi_{n_j}(\omega) = T_{n_j}\left(\omega, \xi_{n_j}(\omega)\right) = f_{T\left(\omega, \xi_{n_j}(\omega)\right)}\left(\lambda_{n_j}\right) \to f_{\xi(\omega)}(1) = \xi(\omega)$. By continuity of $T(\omega, \cdot)$, we have $T\left(\omega, \xi_{n_j}(\omega)\right) \to T\left(\omega, \xi(\omega)\right)$. As $X$ is Hausdorff, we get $T\left(\omega, \xi(\omega)\right) = \xi(\omega)$. ■

The following theorem concerns the Brosowski-Meinardus type theorem on best random approximation and random fixed points.

**Theorem 1.2.2** *Let $X$ be a Banach space. Let $T : \Omega \times X \to X$ be a non-expansive random operator with deterministic fixed point $x$. Assume that $T$ leaves a compact subset $M$ of $X$ as invariant and $P_M(x)$ has the property of contractiveness and joint continuity. Then $x$ has a best random approximation $\xi : \Omega \to M$ which is also a random fixed point of $T$.*

**Proof.** It is easy to see that $P_M(x)$ is nonempty. Let $b \in P_M(x)$. As $T(\omega, \cdot)$ is nonexpansive, so $d(x, M) \leq \|x - T(\omega, b)\| = \|T(\omega, x) - T(\omega, b)\| \leq \|x - b\| = d(x, M)$. So, $\|x - T(\omega, b)\| = d(x, M)$ for each $b \in P_M(x)$. Therefore, $P_M(x)$ is $T(\omega, \cdot)$-invariant for each $\omega \in \Omega$. Also, $P_M(x)$ being a closed subset of a compact subset is compact. Therefore, by Theorem 1.2.1, $T$ has a random fixed point in $P_M(x)$. ■

The following is a random analogue of a result of Dotson [14].

**Theorem 1.2.3** *Let $T : \Omega \times S \to S$ be a weakly continuous nonexpansive* *random operator where $S$ is a nonempty weakly compact subset of a separable Banach space $X$ and suppose there exists a contractive jointly weakly continuous family $F$ of functions associated with $S$. Then $T$ has a random fixed point.*

**Proof.** As in Theorem 1.2.1, let $\lambda_n = \frac{n}{n+1}$, $n = 1, 2, \ldots$ Define mappings $T_n : \Omega \times S \to S$ by $T_n(\omega, x) = f_{T(\omega,x)}(\lambda_n)$. Then for any $\omega \in \Omega$ and $x, y \in S$, we have

$$\begin{aligned} \|T_n(\omega, x) - T_n(\omega, y)\| &= \left\| f_{T(\omega,x)}(\lambda_n) - f_{T(\omega,y)}(\lambda_n) \right\| \\ &\leq \phi(\lambda_n) \, \|T(\omega, x) - T(\omega, y)\| \\ &\leq \phi(\lambda_n) \|x - y\|. \end{aligned}$$

Then by a result in [23], each $T_n$ has a random fixed point $\xi_n$ and hence $T_n(\omega, \xi_n(\omega)) = \xi_n(\omega)$ for each $\omega \in \Omega$. Now for each $n$, define $G_n : \Omega \to WK(S)$ by $G_n(\omega) = \omega\text{-cl}\,\{\xi_i(\omega) : \ n \leq i\}$, where $\omega$-cl$(A)$ denotes the weak closure of $A$ and $WK(S)$ is the set of all nonempty weakly compact subsets of $S$. Define $G : \Omega \to WK(S)$ by $G(\omega) = \displaystyle\bigcap_{n=1}^{\infty} G_n(\omega)$. The arguments similar to those in [10, Theorem 6.3.2] (see also [20]) ensure the existence of a measurable selector $\xi(\omega)$ of $G(\omega)$. We show that $\xi$ is a random fixed point of $T$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in G(\omega)$, therefore there exists a subsequence $\left(\xi_{n_j}(\omega)\right)$ of $(\xi_n(\omega))$ that converges weakly to $\xi(\omega)$; that is, $\xi_{n_j}(\omega) \xrightarrow{\omega} \xi(\omega)$. Since $T_{n_j}\left(\omega, \xi_{n_j}(\omega)\right) = \xi_{n_j}(\omega)$, we have $T_{n_j}\left(\omega, \xi_{n_j}(\omega)\right) \xrightarrow{\omega} \xi(\omega)$. The weak continuity of $T(\omega, \cdot)$ implies $T\left(\omega, \xi_{n_j}(\omega)\right) \xrightarrow{\omega} T\Big(\omega, \xi(\omega)\Big)$ and hence, using the joint weak continuity, we get $T_{n_j}\left(\omega, \xi_{n_j}(\omega)\right) = f_{T\left(\omega, \xi_{n_j}(\omega)\right)}\left(\lambda_{n_j}\right) \xrightarrow{\omega} f_{T(\omega,\xi(\omega))}(1) = T(\omega, \xi(\omega))$.

By the Hausdorff property of the weak topology, we get the required result $T(\omega, \xi(\omega)) = \xi(\omega)$. ■

The following theorem generalizes a result of Habinaik [19].

**Theorem 1.2.4** *Let $X$ be a separable Banach space and $T : \Omega \times X \to X$ a weakly continuous nonexpansive random operator with deterministic fixed point $x$. Assume that $T$ leaves a weakly compact subset $M$ of $X$ invariant, $T(\omega, \cdot)|M$ is compact and $P_M(x)$ has the property of contractiveness and joint weak continuity. Then the point $x$ has a best random approximation $\xi : \Omega \to M$ which is also a random fixed point of $T$.*

**Proof.** The proof is analogous to that of Theorem 1.2.2; apply Theorem 1.2.3 instead of Theorem 1.2.1.

Suppose that $H = \{f_x\}_{x \in S}$ is a family of functions from $[0, 1]$ into $S$ having the property that for each sequence $(\lambda_n)$ in $(0, 1]$, with $\lambda_n \to 1$ as $n \to \infty$, we have

$$f_x(\lambda_n) = \lambda_n x. \tag{*}$$

We observe that $H \subseteq F$ and it has the additional property that it is contractive, jointly continuous and weakly jointly continuous. ■

**Example 1.2.5** Any subspace, a convex set with 0, a star-shaped subset with center 0 and a cone of a normed space have the family of functions associated with them which satisfy condition (*).

If we restrict to the family $H$, then the operators $T_n$ defined by $T_n(\omega, x) = f_{T(\omega,x)}(\lambda_n) = \lambda_n T(\omega, x)$ are random operators because of the randomness of $T : \Omega \times S \to S$, where $(\Omega, \mathcal{A})$ is a measurable space. Hence all the above theorems remain valid for the family $H$ in the context of an arbitrary measurable space $(\Omega, \mathcal{A})$. The following theorem would be a reformulation of Theorem 1.2.3 in this setting. In fact, it removes the conditions of convexity and fixed point property of $S$ and the strict convexity of $X$ required in Theorem 1 of Xu [49].

**Theorem 1.2.6** *Let $(\Omega, \mathcal{A})$ be an arbitrary measurable space and $S$ a weakly compact subset of a separable Banach space $X$. Suppose that $S$ has a family $H$ satisfying condition $(*)$ and $T : \Omega \times S \to S$ is a weakly continuous nonexpansive random operator. Then $T$ has a random fixed point.*

We finally remark that in the absence of the family $F$ or $H$, random fixed points for nonexpansive random operators may not exist as is clear from the following example which also contradicts Theorems 2.2 and 3.2 of Yi and Zhao [51].

**Example 1.2.7** (cf. [26]), Example 3.5). Let $S = \{0, 1\}$ and $(\Omega, 2^{\Omega})$ be a measurable space. Define $T : \Omega \times S \to S$ by

$$T(\omega, 0) = 1 \text{ and } T(\omega, 1) = 0. \tag{**}$$

Then $T$ is a nonexpansive random operator with no random fixed point because if there is any measurable function $\xi : \Omega \to S$ such that $\xi(\omega) = T(\omega, \xi(\omega))$, $\omega \in \Omega$, then either (i) $\xi(\omega) = 0$, $\omega \in \Omega$, or (ii) $\xi(\omega) = 1$,

$\omega \in \Omega$, or (iii) $\xi(\omega) = 0$ for some $\omega \in \Omega$ and $\xi(\omega) = 1$ for some $\omega \in \Omega$ and obviously equation $(**)$ does not hold in all the three cases.

# Chapter 2

# SOME GENERALIZATIONS OF KY FAN'S BEST APPROXIMATION THEOREM

## 2.1 Introduction

In 1969, Ky Fan proved the following best approximation result:

**Theorem A** ([17], Theorem 1). *Let $C$ be a compact convex set in a locally convex Hausdorff topological vector space $X$. If $f : C \to X$ is continuous, then either $f$ has a fixed point or there exist an $x \in C$ and a continuous seminorm $p$ on $X$ such that*

$$p(x - fx) = d_p(fx, C)$$

*where $d_p(fx, C) = inf\{p(fx - y) : y \in C\}$.*

This well-known best approximation theorem due to Ky Fan plays an important role in approximation theory, fixed point theory, nonlinear analysis, game theory and minimax theorems. Among several applications, this result serves as an important tool in ascertaining approximate solutions of systems of equations. It has been extended in various directions by many authors (e.g. see [36] and [43]). Prolla [39] has generalized it for a pair of continuous functions on a normed space while Sehgal and Singh [40] obtained its generalization for continuous multifunctions. Fixed point theorems for multivalued maps and some other related results have been used to prove the existence of best approximation for multivalued maps (see e.g. [22, 36, 38, 39]).

A hyperconvex space is a metric space satisfying a property about the intersection of closed balls. Recently, approximation theory in hyperconvex spaces has been the focus of several researchers. For more information about approximation theory and related concepts in hyperconvex spaces, we refer to [16, 24, 29, 42] where further references are given.

In this chapter, we employ a variant argument, namely; Ky Fan's intersection lemma to establish approximation results for continuous maps and a discontinuous class of multivalued maps, namely, $^*$-nonexpansive maps on compact convex and noncompact convex sets in the settings of metrizable topological vector spaces and hyperconvex spaces.

In Section 2.4, we establish Ky Fan type approximation results in hyperconvex spaces. Hyperconvexity facilitates in obtaining some results of Section 2.3 under weaker assumptions. Also, our results present multivalued analog of some well known approximation theorems for hyperconvex spaces.

In Section 2.2, we recall certain technical preliminaries and establish notational conventions for the sake of completeness.

## 2.2 Preliminaries

Let $X$ denote a topological vector space (TVS, for short). Throughout, we assume that its topology is tacitly generated by an $F$-norm on it; that is, there is a real-valued map, say, $q$ on $X$ such that (i) $q(x) \geq 0$ and $q(x) = 0$ iff

$x = 0$; (ii) $q(x+y) \leq q(x)+q(y)$; (iii) $q(\lambda x) \leq q(x)$ for all $x, y \in X$ and for all scalars $\lambda$ with $|\lambda| \leq 1$; (iv) if $q(x_n) \to 0$, then $q(\lambda x_n) \to 0$ for all scalars $\lambda$; (v) if $\lambda_n \to 0$, then $q(\lambda_n x) \to 0$ for all $x \in X$, where $(\lambda_n)$ is a sequence of scalars. The formula $d(x, y) = q(x - y)$ defines a metric on $X$. A topological vector space $X$ is called *metrizable* if there is a metric on $X$ such that the metric topology coincides with the given topology.

A generalization of the notion of a single-valued nonexpansive selfmap for multivalued maps has been introduced by Husain and Latif [21] as follows.

Let $X$ be a metrizable TVS, $C \subseteq X$ and $T : C \to 2^X$ a multifunction. Then $T$ is called $^*$*-nonexpansive* (cf. [3, 22, 50]) if for all $x, y \in C$ and $u_x \in Tx$ satisfying $d(x, u_x) = d(x, Tx)$, there exists $u_y \in Ty$ satisfying $d(y, u_y) = d(y, Ty)$ such that $d(u_x, u_y) \leq d(x, y)$. Beg, Khan and Hussain [3], Hussain and Khan [22] and Xu [50] have extensively used this concept in their investigations. Recall that $x$ is a fixed point of $T$ if $x \in Tx$.

A multivalued function $T : C \to 2^X$ is *upper semicontinuous* (*usc*) (*lower semicontinuous* (*lsc*)) if $T^{-1}(B) = \{x \in C : Tx \cap B \neq \phi\}$ is closed (open) for each closed (open) subset $B$ of $X$. If $T$ is both usc and lsc, then it is continuous. We denote by $C(X)$, the family of all nonempty closed subsets of $X$ and $H$ denotes the Hausdorff metric on $C(X)$. A map $T : X \to C(X)$ is called *$H$-continuous* if it is continuous as a map from $X$ into the metric space $(C(X), H)$. If $T$ is comapct-valued, then the two notions of continuity are equivalent (see [52]).

The set of best approximations to $x \in X$ from $C$ is a set-valued map defined as $P_C(x) = \{y \in C : d(x, y) = d(x, C)\}$. If $P_C(x) \neq \phi$ (singleton) for each $x \in X$, then $C$ is called a proximinal (Chebyshev) set. In case $P_C(x)$

is single-valued, it is called a *proximity map* (or a *metric projection*) and is denoted by $p$.

Following Xu [50], we define the set (possibly empty),

$$P_T(x) = \{u_x \in Tx : d(x, u_x) = d(x, Tx)\}.$$

In general, $^*$-nonexpansive maps are neither nonexpansive nor continuous as is clear from the following.

**Example 2.2.1** (see also [22, Example 1.1]). Let $T : [0,1] \to 2^{[0,1]}$ be defined by

$$Tx = \begin{cases} \left\{\frac{1}{2}\right\} \quad , & x \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \\ \left[\frac{1}{4}, \frac{3}{4}\right] \quad , & x = \dfrac{1}{2} \end{cases}$$

Then $P_T(x) = \left\{\frac{1}{2}\right\}$ for all $x \in [0,1]$. This implies that $T$ is a $^*$-nonexpansive map. Observe that

$$H(T(1/3), T(1/2)) = H(\{1/2\}, [1/4, 3/4]) = \max\{0, 1/4\} = 1/4 > 1/6 = |1/3 - 1/2|.$$

So $T$ is not a nonexpansive multivalued map. This map is not lsc because if we take $V_{1/4}$ as a small open neighborhood of $1/4$, then the set

$$T^{-1}(V_{1/4}) = \left\{x \in [0,1] : Tx \cap V_{1/4} \neq \phi\right\} = \left\{\frac{1}{2}\right\}$$

is not open. Hence $T$ is not continuous. Note that $1/2$ is a fixed point of $T$.

A mapping $f : C \to X$ is called a selector of the map $T : C \to 2^X$ if $f(x) \in Tx$. For $x \in X$, the *inward set*, $I_C(x)$, of $C$ at $x$ is defined by $I_C(x) = \{x + r(u - x) \in X : u \in C, \quad r > 0\}$. The closure of $I_C(x)$ is denoted by $\overline{I_C(x)}$.

For a finite subset $\{x_1, \ldots, x_n\}$ of a TVS $X$, we write the *convex hull* of $\{x_1, \ldots, x_n\}$ as

$$Co\{x_1, \ldots, x_n\} = \left\{\sum_{i=1}^{n} \alpha_i x_i : \quad 0 \leq \alpha_i \leq 1, \quad \sum_{i=1}^{n} \alpha_i = 1\right\}.$$

The following result known as Ky Fan's intersection Lemma [17] is needed.

**Theorem B** *Let $C$ be a subset of a TVS $X$ and $F : C \to 2^X$ a closed-valued map such that $Co(x_1, \ldots, x_n) \subseteq \bigcup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1, \ldots, x_n\}$ of $C$. If $F(x_0)$ is compact for at least one $x_0$ in $C$, then $\bigcap_{x \in C} F(x) \neq \phi$.*

*Let $C$ be a convex subset of metrizable TVS $X$ and $g : C \to C$ a continuous map. Then $g$ is said to be*

*(i) <u>almost affine</u> if*

$$d\Big(g(rx_1 + (1-r)x_2), y\Big) \leq rd(gx_1, y) + (1-r)d(gx_2, y),$$

*(ii) <u>almost quasi-convex</u> if*

$$d\Big(g(rx_1 + (1-r)x_2), y\Big) \leq \max\{d(gx_1, y), d(gx_2, y)\},$$

*where $x_1, x_2 \in C$, $y \in X$ and $0 < r < 1$.*

It is easy to see that (i) implies (ii), but not conversely, in general (see also [44] for related concepts).

A metric space $(Y, d)$ is said to be *hyperconvex* if $\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \phi$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of all closed balls in $Y$ for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ (see e.g. [2]). An *admissible* subset of a hyperconvex space $Y$ is a set of the form $\bigcap_{\alpha} B(x_\alpha, r_\alpha)$, where $\{B(x_\alpha, r_\alpha)\}$ is a family of closed balls centered at the points $x_\alpha \in Y$ with respective radii $r_\alpha$. It is well-known that an admissible subset of a hyperconvex space is itself hyperconvex (see e.g. [24]). A subset $E$ of a metric space $Y$ is said to be *externally hyperconvex* (relative to $Y$) if for a given family $\{x_\alpha\}$ of points in $Y$ and a family $\{r_\alpha\}$ of real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $d(x_\alpha, E) \leq r_\alpha$, we have $\bigcap_{\alpha} B(x_\alpha, r_\alpha) \cap E \neq$

$\phi$. It is shown in [2] that an admissible subset of a hyperconvex space $Y$ is externally hyperconvex relative to $Y$, and externally hyperconvex subsets of $Y$ are proximinal in $Y$. Thus, if $E$ is externally hyperconvex in $Y$ and $x \in Y$, then there is $h \in E$ such that $d(x, h) = d(x, E)$. For more information on externally hyperconvex spaces, we refer to [24].

A subset $E$ of a metric space $Y$ is said to be *weakly externally convex* (relative to $Y$) if $E$ is externally hyperconvex relative to $E \cup \{z\}$ for each $z \in Y$. More precisely, given any family $\{x_\alpha\}$ of points in $Y$ all but at most one of which lies in $E$, and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ $\Big($ with $d(x_\alpha, E) \leq r_\alpha$ if $x_\alpha \notin E\Big)$ implies that $\bigcap_\alpha B(x_\alpha, r_\alpha) \cap E \neq \phi$ (see [16] for more details).

In what follows we use $E(Y)$ to denote the family of all bounded subsets of $Y$ which are externally hyperconvex.

## 2.3 Approximation Results

Khan, Thaheem and Hussain [27] have recently established the following pair of Prolla type approximation theorems on the basis of Ky Fan's intersection lemma (i.e. Theorem B) and the arguments used by Carbone [11]. The proofs of these theorems can be found in Theorems 3.3.1 and 3.3.2.

**Theorem C** ([27], Theorem 3.1). *Let $C$ be a nonempty compact convex subset of a metrizable TVS $X$ and $g : C \to C$ a continuous almost quasi-convex onto function. If $f : C \to X$ is a continuous function, then there exists $y \in C$ such that $d(gy, fy) = d(fy, C)$.*

**Theorem D** ([27], Theorem 3.2). *Let $C$ be a nonempty convex subset of a metrizable TVS $X$ and $g : C \to C$ a continuous almost quasi-convex onto* *function. Suppose $f : C \to X$ is a continuous function. If $C$ has a nonempty compact convex subset $B$ such that the set*

$$D = \{y \in C : d(fy, gy) \leq d(fy, gx) \text{ for all } x \in B\}$$

*is compact, then there exists $y \in D$ such that $d(fy, gy) = d(fy, C)$.*

The "selections" have been studied and used in a number of disciplines over the last fifty years. Recently, Agarwal and O'Regan [1], Espinola, Kirk and Lopez [16], Hussain and Khan [22], Khamsi, Kirk and Martinez-Yanez [24] have utilized "selections" to obtain fixed point and approximation results for multivalued maps. In general, a nonexpansive multivalued map does not admit a single-valued nonexpansive selection. However, $^*$-nonexpansive maps and hyperconvex spaces in many situations share this property.

We establish here new Prolla type approximation results by using continuous selectors of $^*$-nonexpansive maps on metrizable TVS and hyperconvex spaces. Our results contain among others, the Ky Fan type approximation theorems as a special case.

The following result provides a generalized Prolla type best approximation theorem for $^*$-nonexpansive maps.

**Theorem 2.3.1** *Let $C$ be a nonempty compact convex subset of a uniformly convex metrizable TVS $X$ and $g : C \to C$ a continuous almost quasi-convex onto map. If $T : C \to 2^X$ is a closed convex valued $^*$-nonexpansive mapping, then $T$ possesses a nonexpansive selector $f$ such that $d(gy, fy) = d(fy, C)$ for some $y \in C$. Further,*

*(i) If $T : C \to 2^C$, then $y$ is a coincidence point of $g$ and $T$, that is, $gy \in Ty$.*

*(ii) If $d(fy, pfy) = d(Ty, C)$, then $d(gy, Ty) = d(Ty, C)$, where $p$ is the proximity map of $X$ onto $C$.*

**Proof.** A closed convex set in a uniformly convex metric linear space is Chebyshev, so $Tx$ is a Chebyshev subset of $X$ for each $x \in C$. Thus for each $x \in C$, there is unique $u_x \in Tx$ such that $\{u_x\} = P_T(x) \in Tx$. Since $T$ is $^*$-nonexpansive, therefore for each $x, y \in C$, we get

$$d(P_T(x), P_T(y)) = d(u_x, u_y) \leq d(x, y).$$

This implies that $P_T : C \to X$ is a nonexpansive selector of $T$ (i.e. $P_T(x) \in Tx$). By Theorem C, there exists $y \in C$ such that

$$d(gy, P_T(y)) = d(P_T(y), C) \tag{2.3.1}$$

This proves the first part of the theorem.

To prove (i), we observe that $d(P_T(y), C) = 0$ implies that $gy = P_T(y) \in Ty$ as desired.

To prove (ii), we note that equation (2.3.1) and the assumption that $d(fy, pfy) = d(Ty, C)$ imply

$$d(gy, Ty) \leq d(gy, fy) = d(fy, C) = d(fy, pfy) = d(Ty, C) \leq d(gy, Ty).$$

Therefore, $d(gy, Ty) = d(Ty, C)$. ■

Next, we obtain a version of Theorme 2.3.1 without the compactness of $C$.

**Theorem 2.3.2** *Let $C$ be a nonempty convex subset of a uniformly convex metrizable TVS $X$, $g : C \to C$ continuous almost quasi-convex onto function and $T : C \to 2^X$ a closed convex valued $^*$-nonexpansive mapping. Assume that $C$ has a nonempty compact convex subset $B$ such that*

*(i) $D = \{z \in C : d(Tz, gz) \leq d(Tz, gx)$ for all $x \in B\}$ is compact.*

*(ii) for each $z \in C$, $d(u_z, gx) \leq d(Tz, gx)$, where $z \in C$ and satisfies $d(u_z, gz) \leq d(u_z, gx)$ for each $x \in C$.*

(Here $u_z$ denotes the unique best approximation of $z$ from $Tz$). Then there exists $y \in D$ such that $d(gy, Ty) = d(Ty, C)$.

**Proof.** As in the proof of Theorem 2.3.1, $P_T : C \to X$ is a nonexpansive selector of $T$. Define $E = \{y \in C : d(P_T(y), gy) \leq d(P_T(y), gx)$ for each $x \in B\}$. As both $P_T$ and $g$ are continuous, so $E$ is a closed subset of $C$. Let $y \in E$. Then for each $x \in B$, we have (by (i))

$$\begin{aligned} d(Ty, gy) &\leq d(P_T y, gy) \leq d(P_T y, gx) \\ &= d(u_y, gx) \leq d(Ty, gx). \end{aligned}$$

This implies that $y \in D$. Thus $P_T$ satisfies all the conditions of Theorem D and hence there exists $y \in D$ such that

$$d(gy, P_T y) = d(P_T y, C) \tag{2.3.2}$$

From (2.3.2) and the hypotheses $d(P_T z, gx) \leq d(Tz, gx)$, we get the inequality

$$\begin{aligned} d(gy, Ty) &= d(gy, P_T y) = d(P_T y, C) \leq d(P_T y, gx) \\ &\leq d(Ty, gx) \end{aligned}$$

for all $x \in C$.

As $g$ is onto, so $d(gy, Ty) = d(Ty, C)$. ■

**Remark 2.3.3** *(i) In case the map $T$ is $H$-continuous instead of being $^*$-nonexpansive, the conclusions of Theorems 2.3.1 and 2.3.2 hold (the same proofs carry over).*

*(ii) If we consider $T : C \to 2^C$ in Theorem 2.3.2, then $y$ becomes a coincidence point of $g$ and $T$.*

*(iii) All the results obtained so far hold good when $X$ is a Fréchet space.*

## 2.4 Approximation in Hyperconvex Spaces

We begin with an analog of Theorem 2.3.1 under weaker conditions. This also gives a multivalued extension of results of Sine [42, Corollary 12] and Espinola, Kirk and López [16, Theorem 5.4].

**Theorem 2.4.1** *Let $C$ be a nonempty compact convex subset of a hyperconvex metrizable TVS $X$, $g : C \to C$ a continuous almost quasi-convex onto map and $T : C \to 2^X$. Suppose that either of the following conditions (a), (b) and (c) holds:*

*(a) $T$ is $^*$-nonexpansive and for each $x \in C$, $Tx$ is externally hyperconvex.*

*(b) $T$ is continuous and $Tx$ is bounded and externally hyperconvex for each $x \in C$.*

*(c) $X$ has unique metric segments and $T$ is closed-valued $^*$-nonexpansive.*

*Then $T$ possesses a continuous selector $f$ such that $d(gy, fy) = d(fy, C)$ for some $y \in C$. Further,*

*(i) if $T : C \to 2^C$, then $y$ is a coincidence point of $g$ and $T$;*

*(ii) if $d(fy, pfy) = d(Ty, C)$, then $d(gy, Ty) = d(Ty, C)$, where $p$ is the proximity map of $X$ onto $C$.*

**Proof.** (a) Each $Tx$ being nonempty externally hyperconvex is proximinal, therefore $P_T(x)$ is nonempty for each $x \in C$ and $P_T(x) = B(x, r) \cap Tx$, where $r = d(x, Tx)$. $P_T(x)$ being the intersection of admissible and externally hyperconvex sets is externally hyperconvex for each $x \in C$ [24, Lemma 2]. So, $P_T : C \to E(X)$ is nonexpansive by the $^*$-nonexpansive axiom of $T$. Thus by

[24, Corollary 1], $P_T$ has a nonexpansive selector $f : C \to X$ which is also a selector of $T$. By Theorem C, there exists $y$ in $C$ such that $d(gy, fy) = d(fy, C)$. This proves the first part of the result.

The proof for (i) is simple and we omit it.

To prove (ii), we note that the equality $d(gy, fy) = d(fy, C)$ and the hypotheses imply $d(gy, Ty) \leq d(gy, fy) = d(fy, C) = d(fy, pfy) = d(Ty, C) \leq d(gy, Ty)$.

(b) The selection theorem of [24, Theorem 1] implies that $T$ has a continuous selection $f : C \to X$. Then, by Theorem C, there exists $y \in C$ such that $d(gy, fy) = d(fy, C)$ and following the arguments similar to those in (a), we get the proof for (b).

(c) We observe that a hyperconvex metric space with unique metric segments is a complete $\mathbb{R}$-tree [29, Theorem 3.2]. Further, a closed subtree of a complete $\mathbb{R}$-tree is Chebyshev [29, pp. 70–71]. Thus, $P_T(x)$ in $Tx$ is unique for each $x$ in $C$ and hence $P_T : C \to X$ is a nonexpansive selector of $T$. So, the result follows from (a). ■

The following theorem is a multifunction analog of the results of Sine [42] and Espinola, Kirk and López [16] for hyperconvex normed spaces.

**Theorem 2.4.2** *Suppose that $C$ is a nonempty convex and weakly externally hyperconvex subset of a hyperconvex normed space $X$ and $T : C \to 2^X$ satisfies either of the conditions (a), (b) and (c) of Theorem 2.4.1. Let $M$ and $K$ be compact subsets of $C$ with $M$ being convex. If for each $x$ in $C \backslash K$, $x \notin P_M(Tx)$, then $T$ possesses a continuous selector $f$ such that $d(y, fy) = d(fy, \overline{I_C(y)})$ for some $y \in K$. If, in addition, $d(fy, pfy) = d(Ty, \overline{I_C(y)})$, where $p$ is the proximity map of $X$ onto $C$, then $d(y, Ty) = d(Ty, \overline{I_C(y)})$.*

**Proof.** As in the proof of Theorem 2.4.1, $T$ has a continuous selector $f$ in all the cases (a), (b) and (c). Thus, $P_M(fx) \subseteq P_M(Tx)$ for each $x \in C$. So, by assumption $x \notin P_M(f(x))$ for each $x \in C \setminus K$. Thus, by Theorem 1(i) of Park [37], there is $y$ in $K$ such that $d(y, fy) = d(fy, (\overline{I_C(y)}))$. Since $C$ being weakly externally hyperconvex is proximinal, therefore $pfy$ is a nonempty subset of $C$. By hypothesis, $d(fy, pfy) = d(Ty, (\overline{I_C(y)}))$, and hence we get

$$\begin{aligned} d(y, Ty) &\leq d(y, fy) = d(fy, (\overline{I_C(y)})) \leq d(fy, C) \leq d(fy, pfy) \\ &= d(Ty, (\overline{I_C(y)})) \leq d(y, Ty). \end{aligned}$$

■

Finally, since compact hyperconvex subspaces have the fixed point property for continuous single-valued mappings [16], the selection theorem [24, Theorem 1] and Theorem 4.2 of [16] yield the following best approximation result for compact weakly externally hyperconvex set in hyperconvex metric spaces which is, in fact, a multifunction analog of Theorem 5.4 of [16].

**Theorem 2.4.3** *Suppose that $C$ is a nonempty compact weakly externally hyperconvex subset of a hyperconvex space $X$ and $T$ satisfies either of the conditions (a), (b) and (c) of Theorem 2.4.1. Then $T$ possesses a continuous selector $f$ such that $d(y, fy) = d(fy, C)$ for some $y \in C$. If, in addition, $d(fy, pfy) = d(fy, C)$, then $d(y, Ty) = d(Ty, C)$, where $p$ is the proximity map of $X$ onto $C$.*

**Remark 2.4.4** If we consider $T : C \to 2^C$ in Theorem 2.4.3, then we obtain the following fixed point result (Corollary 2.4.5) for $^*$-nonexpansive and continuous maps which extends several well known results such as Theorem 3.2 of [21], Theorem 2 of Xu [50], Corollary 4 of [24] and Corollaries 3.3 and 3.4 of Kirk [29].

**Corollary 2.4.5** *Suppose that $C$ is a nonempty compact weakly externally hyperconvex subset of a hyperconvex space $X$ and $T : C \to 2^C$ satisfies either of the conditions (a), (b) and (c) of Theorem 2.4.1. Then $T$ has a fixed point.*

# Chapter 3

# A STOCHASTIC VERSION OF FAN'S BEST APPROXIMATION THEOREM

## 3.1 Introduction

A lot of work has been done on the existence of best approximation for continuous and nonexpansive mappings on Hilbert spaces, Banach spaces and locally convex topological vector spaces. These results include both single and multivalued maps. In general, fixed point theorems and the related techniques have been used to prove the results about best approximation. We refer to [11, 17, 27, 39, 41] and references therein.

In this chapter, we generalize Prolla's main result by considering a continuous function and the other one being a continuous almost quasi-convex onto function on a suitable subset of a metrizable topological vector space, using Ky Fan's intersection lemma [17] (see Theorem B) as a main tool. Stochastic versions of our results are established as well. As a consequence, a stochastic generalization of the celebrated Fan's best approximation theorem (Theorem

A) follows.

In Section 3.3, we prove some approximation results for single-valued continuous quasi-convex mappings on a compact as well as on a noncompact subset of a metrizable topological vector space.

In Section 3.4, we present random versions of the results in Section 3.3. Section 3.2 deals with certain technical preliminaries and establishes notational conventions. Even though some of the concepts are standard, they are included here to facilitate reading.

## 3.2 Preliminaries

Let $X$ denote a topological vector space. We denote by $2^X$, $C(X)$ and $CK(X)$ the families of all nonempty, nonempty closed and nonempty convex compact subsets of $X$.

Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a $\sigma$-algebra of subsets of $\Omega$. Let $P(Z)$ be a collection of subsets of a set $Z$. Denote by $\hat{N}$ the set of all infinite sequences of positive integers and by $\hat{N}_0$, the set of all finite sequences of positive integers. A subset $A$ of $Z$ is said to be obtained from $P(Z)$ by *Souslin operation* if there is a map $k : \hat{N}_0 \to P(Z)$ such that $A = \bigcup_{x \in \hat{N}} \bigcap_{n=1}^{\infty} (k(r|_n)$, where $r|_n$ denotes the first $n$ elements of the finite sequence $r \in \hat{N}$. Note that the union in the Souslin operation is uncountable. So, if $P(Z)$ is a $\sigma$-algebra, then $A$ may be outside $P(Z)$. If $P(Z)$ is closed under the Souslin operation, then it is called a *Souslin family*. For more details about Souslin family we refer to Shahzad [41] and Wagner [48].

Let $T : \Omega \to 2^X$ be a multivalued mapping. The set

$$Gr(T) = \{(\omega, x) \in \Omega \times X : x \in T(\omega)\}$$

is called the *the graph* of $T$.

A mapping $T : \Omega \to 2^X$ is said to be *measurable* (respectively, *weakly measurable*) if $T^{-1}(B) \in \Sigma$ for each closed (respectively, open) subset $B$ of $X$, where

$$T^{-1}(B) = \{\omega \in \Omega : T(\omega) \cap B \neq \phi\}.$$

It is known that the measurability of $T : \Omega \to 2^X$ implies the weak measurability but not conversely, in general.

Let $Y, Z$ be two metric spaces. A function $f : \Omega \times Y \to Z$ is said to be a *Caratheodory function* if for each $y \in Y$, $T(\cdot, y)$ is measurable and for each $\omega \in \Omega$, $T(\omega, \cdot)$ is continuous.

Random operators with stochastic domain have been studied by Engl [15] and Shahzad [41].

Following Engl [15] and Papageorgiou [35], we say that a mapping $T : \Omega \to 2^X$ is *separable* if there exists a countable set $D \subseteq X$ such that for all $\omega \in \Omega$, $cl(D \cap T(\Omega)) = T(\omega)$. For instance, if $T$ has closed, convex and solid (that is, nonempty interior) values, then $T$ is separable. Further, it is clear from the definition of separability that $T$ has closed values.

Let $F : \Omega \to C(X)$ be a weakly measurable mapping. A mapping $T : Gr(F) \to 2^X$ is called a *multivalued random operator* with stochastic domain $F(\cdot)$ if for all $x \in X$ and all $U \subseteq X$ open, $\{\omega \in \Omega : T(\omega, x) \cap U \neq \phi, \quad x \in F(\omega)\} \in \Sigma$.

Let $F : \Omega \to C(X)$ be a weakly measurable mapping. A random operator $T : Gr(F) \to X$ with stochastic domain $F(\cdot)$ is called a *random contraction* if

$T(\omega, \cdot)$ is a contraction on $F(\omega)$ for all $\omega \in \Omega$.

THEOREM E. ([31], Theorem 1). *Let $C$ be a nonempty convex subset of a Hausdorff TVS $X$ and $A \subseteq C \times C$ such that*

*(a) for each $x \in C$, the set $\{y \in C : (x, y) \in A\}$ is closed in $C$;*

*(b) for each $y \in C$, the set $\{x \in C : (x, y) \notin A\}$ is convex or empty;*

*(c) $(x, x) \in A$ for each $x \in C$;*

*(d) $C$ has a nonempty compact convex subset $X_0$ such that the set $B = \{y \in C : (x, y) \in A \text{ for all } x \in X_0\}$ is compact.*

*Then there exists a point $y_0 \in B$ such that $C \times \{y_0\} \subset A$.*

A random operator $f : \Omega \times C \to X$ is continuous (almost affine, almost quasi-convex) if for each $\omega \in \Omega$, the map $f(\omega, \cdot) : C \to X$ is so. For some concepts and terminology used here, we refer to Chapter 2.

## 3.3 Approximation in Metrizable Topological Vector Spaces

We begin with the following theorem which generalizes the main result of Prolla [39] to a wider class of functions defined on a subset of a metrizable TVS with its proof based on Ky Fan's intersection lemma (Theorem B). This result also extends Theorem 1 of Carbone [11] and partially extends Theorem 2.1 in [36] (see also [38]).

**Theorem 3.3.1** *Let $C$ be a nonempty compact convex subset of a metrizable TVS $X$ and $g : C \to C$ a continuous almost quasi-convex onto function.*

*If $f : C \to X$ is a continuous function, then there exists $y \in C$ such that $d(gy, fy) = d(fy, C)$.*

**Proof.** For each $z \in C$, define

$$F(z) = \{y \in C : d(gy, fy) \leq d(gz, fy)\}.$$

Since $f$ and $g$ are continuous, therefore for each $z \in C$, $F(z)$ is a closed set and hence a compact subset of $C$.

Let $\{x_1, \ldots, x_n\}$ be a finite subset of $C$. Then, $Co(x_1, \ldots, x_n) \subseteq \bigcup_{i=1}^{n} F(x_i)$. If this is not the case, then there is some $u$ in $Co(x_1, \ldots, x_n)$ such that $u \notin \bigcup_{i=1}^{n} F(x_i)$. Now $u = \sum_{i=1}^{n} \alpha_i x_i$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$ and as $u \notin \bigcup_{i=1}^{n} F(x_i)$, so $d(gx_i, fu) < d(gu, fu)$ for all $i = 1, 2, \ldots, n$. Since $g$ is almost quasi-convex, therefore

$$d(gu, fu) = d\left(g\left(\sum_{i=1}^{n} \alpha_i x_i\right), fu\right) \leq \max_i d(gx_i, fu) < d(gu, fu)$$

which is impossible. Then, by Theorem B, $\bigcap_{x \in C} F(x) \neq \phi$ and hence there is $y \in \bigcap_{x \in C} F(x)$ so that, for all $x \in C$,

$$d(gy, fy) \leq d(gx, fy).$$

Since $g$ is onto, we get $d(gy, fy) \leq d(z, fy)$ for all $z \in C$ and hence $d(gy, fy) = d(fy, C)$. ■

The compactness of $C$ in Theorem 3.3.1 can be replaced by a weaker condition to obtain the following generalization of Theorem 2 of Carbone [11].

**Theorem 3.3.2** *Let $C$ be a nonempty convex subset of a metrizable TVS $X$ and $g : C \to C$ a continuous almost quasi-convex onto function. Suppose*

*$f : C \to X$ is a continuous function. If $C$ has a nonempty compact convex subset $B$ such that the set*

$$D = \{y \in C : d(fy, gy) \leq d(fy, gx) \text{ for all } x \in B\}$$

*is compact, then there exists an element $y \in D$ such that $d(fy, gy) = d(fy, C)$.*

**Proof.** Let $A = \{(x, y) \in C \times C : d(fy, gy) \leq d(fy, gx)\}$. Obviously, $(x, x) \in A$ for all $x \in C$. By the continuity of $f$ and $g$, the set $\{y \in C : (x, y) \in A\}$ is closed in $C$ for each $x \in C$. The set

$$K = \{x \in C : (x, y) \notin A\} = \{x \in C : d(fy, gy) > d(fy, gx)\}$$

is convex. Indeed, suppose $x_1, x_2 \in K$. Then $d(gx_1, fy) < d(fy, gy)$ and $d(gx_2, fy) < d(fy, gy)$. Since $g$ is almost quasi-convex, we have for $0 < \lambda < 1$,

$$\begin{aligned} d(g(\lambda x_1 + (1-\lambda)x_2), fy) &\leq \max\{d(gx_1, fy), d(gx_2, fy)\} \\ &< d(fy, gy). \end{aligned}$$

This implies that $\lambda x_1 + (1-\lambda)x_2 \in K$.

By Theorem E, there exists $y \in D$ such that $C \times \{y\} \subset A$. That is, $d(fy, gy) \leq d(fy, gx)$ for all $x \in C$. As $g$ is onto, so $d(fy, gy) = d(fy, C)$ for some $y \in B$. ■

**Remark 3.3.3** (i) If we consider $f : C \to C$ in Theorems 3.3.1 and 3.3.2, then $y$ becomes a coincidence point of $f$ and $g$ (that is, $fy = gy$).

(ii) All the results obtained so far trivially hold when $X$ is a Fréchet space.

## 3.4 Random Approximation

In this section we establish the random versions of Theorems 3.3.1 and 3.3.2 which in turn extend Theorem 5 of [8] and Theorem 5 of [41] to the general framework of metrizable topological vector spaces.

**Theorem 3.4.1** *Let $C$ be a compact and convex subset of a complete metrizable TVS $X$ and $g : \Omega \times C \to C$ a continuous almost quasi-convex and onto random operator. If $T : \Omega \times C \to X$ is a continuous random operator, then there exists a measurable map $\xi : \Omega \to C$ satisfying*

$$d(g(\omega, \xi(\omega)), T(\omega, \xi(\omega))) = d(T(\omega, \xi(\omega)), C)$$

*for each $\omega \in \Omega$.*

**Proof.** Let $F : \Omega \to 2^C$ be defined by

$$F(\omega) = \{x \in C : d(g(\omega, x), T(\omega, x)) = d(T(\omega, x), C)\}.$$

By Theorem 3.3.1, $F(\omega) \neq \phi$ for all $\omega \in \Omega$. Also, $F(\omega)$ is compact for each $\omega \in \Omega$. Let $G$ be a closed subset of $C$. Put

$$L(G) = \bigcap_{n=1}^{\infty} \bigcup_{x \in D_n} \Big\{\omega \in \Omega : d(g(\omega, x), T(\omega, x)) < d(T(\omega, x), C) + \frac{1}{n}\Big\},$$

where $D_n = \left\{x \in D : d(x, G) < \dfrac{1}{n}\right\}$.

Note that the functions $p : \Omega \times C \to \mathbb{R}^+$ and $q : \Omega \times C \to \mathbb{R}^+$ defined by $p(\omega, x) = d(g(\omega, x), T(\omega, x))$ and $q(\omega, x) = d(T(\omega, x), C)$ are measurable in $\omega$ and continuous in $x$ (see [41, Theorem 5]). Following arguments similar to those in the proof of Theorem 5 of [8], we can show that $F$ is measurable. Applying a selection theorem due to Kuratowski and Nardzewski [30] we get a measurable map $\xi : \Omega \to C$ such that $\xi(\omega) \in F(\omega)$ for all $\omega \in \Omega$. The result now follows from the definition of $F(\omega)$. ∎

**Definition 3.4.2** Let $(X, d_1)$ and $(Y, d_2)$ be two metric spaces. The pair of metric spaces $(X, Y)$ is said to have the *Kirzbraun property* or *property* $(K)$ according to Shahzad [41] if for all choices $x_i \in X$, $y_i \in Y$ and $\gamma_i > 0$, $i \in I$

($I$ an arbitrary index set) such that the intersection of the balls $B(y_i, \gamma_i)$ in $X$ is nonempty and $d_2(y_i, y_j) \leq d_1(x_i, x_j), \quad i, j \in I$, then the intersection of the balls $B(y_i, \gamma_i)$ in $Y$ is also nonempty.

We need the following result of Shahzad [41, Theorem 1].

**Theorem 3.4.3** *Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a Souslin family. Let $X$ and $Y$ be separable complete metric spaces such that the pair $(X, Y)$ has property $(K)$ and $F : \Omega \to 2^X$ a separable weakly measurable function. Then every random contraction $f : Gr(F) \to Y$ with stochastic domain $F(\cdot)$ can be extended to a random contraction defined on $\Omega \times X$.*

**Remark 3.4.4** The conclusion of Lemma 6 of Engl [15] remains valid for separable complete metric spaces (cf. [41], p. 442).

**Theorem 3.4.5** *Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a Souslin family and $X$ a separable complete metrizable TVS. Assume that $F : \Omega \to 2^X$ is a separable weakly measurable convex-valued multifunction and $f : Gr(F) \to X$ is a random contraction with stochastic domain $F(\cdot)$. If $g : Gr(F) \to X$ is a continuous almost quasi-convex onto random operator with stochastic domain $F(\cdot)$ such that $g(\omega, x) \in F(\omega)$ for all $(\omega, x) \in Gr(F)$. Suppose that $G_0 : \Omega \to CK(X)$ is a measurable multifunction with $G_0(\omega) \subseteq F(\omega)$ for all $\omega \in \Omega$ such that for a weakly measurable multifunction $D$,*

$$D(\omega) = \{y \in F(\omega) : d(f(\omega, y), g(\omega, y)) \leq d(f(\omega, y), g(\omega, x)) \text{ for all } x \in G_0(\omega)\}$$

*is compact for each $\omega \in \Omega$. If the pair $(X, X)$ has property $(K)$, then there exists a measurable map $\xi : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi(\omega) \in D(\omega)$ and*

$$d(f(\omega), \xi(\omega)), g(\omega, \xi(\omega))) = d(f(\omega, \xi(\omega)), F(\omega)).$$

**Proof.** By Theorem 3.4.3, we get a random contraction $\hat{f} : \Omega \times X \to X$. Let $H : \Omega \to 2^X$ be defined by

$$H(\omega) = \left\{ x \in D(\omega) : d(g(\omega, x), \hat{f}(\omega, x)) = d(\hat{f}(\omega, x), F(\omega)) \right\}.$$

By Theorem 3.3.2, $H(\omega) \neq \phi$ for each $\omega \in \Omega$. Define maps $h, k : \Omega \times X \to \mathbb{R}^+$ by $h(\omega, x) = d(\hat{f}(\omega, x), F(\omega))$ and $k(\omega, x) = d(\hat{f}(\omega, x), g(\omega, x))$. Obviously $h$ is continuous and by [15, Lemma 6], $h$ is measurable in $\omega$ (see Remark 3.4.4), so $h(\cdot, \cdot)$ is a Caratheodory function. Similarly $k(\cdot, \cdot)$ is also a Caratheodory function. By the continuity of functions involved, $H(\omega)$ is closed for each $\omega \in \Omega$.

Define $\phi(\omega, x) = h(\omega, x) - k(\omega, x)$. Clearly, $\phi(\cdot, \cdot)$ is jointly measurable. Observe that

$$Gr(H) = Gr(F) \cap \{(\omega, x) \in \Omega \times X : \phi(\omega, x) = 0\} \in \Sigma \times B(X).$$

Since $\Sigma$ is a Souslin family, therefore by [48, Theorem 4.2], $H(\cdot)$ is weakly measurable. By the selection theorem in [30], $H(\cdot)$ has a measurable selector $\xi : \Omega \to X$. Consequently, $\xi(\omega) \in D(\omega)$ and

$$d(f(\omega, \xi(\omega)), g(\omega, \xi(\omega))) = d(f(\omega, \xi(\omega)), F(\omega))$$

for each $\omega \in \Omega$. ■

An immediate consequence of the above theorem when the underlying domain of the maps $f$ and $g$ is not varying stochastically is presented below; our result generalizes Corollary 2 in [7] to metrizable TVS.

**Corollary 3.4.6** *Let $(\Omega, \Sigma)$ and $X$ be as in Theorem 3.4.5 and $C$ a nonempty convex subset of $X$. Assume that $f : \Omega \times C \to X$ is a random contraction and $g : \Omega \times C \to C$ is a continuous almost quasi-convex onto random operator. Let*

*$X_0$ be a nonempty compact convex subset of $C$ and $K$ be a nonempty compact subset of $C$. If for each $y \in C \setminus K$, there exists $x \in X_0$ such that*

$$d(g(\omega, x), f(\omega, y)) < d(g(\omega, y), f(\omega, y)),$$

*then there exists a measurable mapping $\xi : \Omega \to K$ satisfying*

$$d(g(\omega, \xi(\omega)), f(\omega, \xi(w)) = d(f(\omega, \xi(\omega)), C)$$

*for each $\omega \in \Omega$.*

**Remark 3.4.7** Theorem 3.4.5 extends Corollary 3.3 [6], Theorem 1 [7], Theorem 5 [8], Theorem 4 [35] and Theorem 5 [41] to the general framework of metrizable topological vector spaces.

# Chapter 4

# ON SOME PROPERTIES OF BANACH OPERATORS

## 4.1 Introduction

This chapter is a continuation of our earlier work [47] on Banach operators. Banach operators are generalizations of contraction maps and play an important role in the fixed point theory; their consideration goes back to Cheney and Goldstein [13] in the study of proximity maps on convex sets (see [34] and references therein).

In [47], we established some decompositional properties of a normed space using Banach operators. We showed that if $\alpha$ is a linear Banach operator on a normed space $X$, then $N(\alpha-1) = N((\alpha-1)^2)$, $N(\alpha-1)\cap R(\alpha-1) = (0)$ and in case $X$ is finite-dimensional, we get the decomposition $X = N(\alpha-1)\oplus R(\alpha-1)$, where $N(\alpha - 1)$ and $R(\alpha - 1)$ denote the null space and the range space of $(\alpha - 1)$, respectively and 1 denotes the identity operator on $X$. In Proposition 2.3 of [47], we proved a decompositional property of a general bounded linear operator on a Hilbert space; namely, if $\alpha$ is a bounded linear operator on a Hilbert space $H$ such that $\alpha$ and $\alpha^*$ (adjoint of $\alpha$) have common fixed points, then $N(\alpha - 1) + R(\alpha - 1)$ is dense in $H$.

In this chapter we mainly focus on Banach operators on a Hilbert space and establish a stronger form of the decomposition results of [47]. We show (Proposition 4.2.1) that if $\alpha$ is a bounded linear Banach operator on a Hilbert space $H$ such that $\alpha$ and $\alpha^*$ have common fixed points, then $H$ admits the decomposition $H = N(\alpha - 1) \oplus M$, where $M = \overline{R(\alpha - 1)}$, ($\overline{R(\alpha - 1)}$ denotes the closure of $(\alpha - 1)$). Further, with the same assumptions as in Proposition 4.2.1, we show that there exists a largest projection $P$ on $H$ such that $\alpha(Px) = Px$ for all $x \in H$.

As in [47], we also study the operator equation $\alpha + c\alpha^{-1} = \beta + c\beta^{-1}$ for a pair of invertible bounded linear multiplicative Banach operators $\alpha$ and $\beta$ on a normed algebra with identity for an appropriate real or complex number $c$. We show (Proposition 4.2.3) that if $\alpha(x) + c\alpha^{-1}(x) = \beta(x) + c\beta^{-1}(x)$ for all $x \in X$, where $c$ is a real or complex number such that $|c| \geq 1$, $\|\alpha\|^2 \leq \frac{|c|}{2}$ , $\|\beta\|^2 \leq \frac{|c|}{2}$ and if $\beta$ is inner, then $\alpha = \beta$. We briefly recall that this operator equation has been extensively studied for automorphisms on von Neumann algebras. We refer to [12, 45, 46] for more details about this operator equation.

## 4.2 The Results

**Proposition 4.2.1** *Let $\alpha$ be a bounded linear Banach operator on a Hilbert space $H$ such that $\alpha$ and $\alpha^*$ have common fixed points. Then the following hold:*

*(i)* $N(\alpha - 1) \cap R(\alpha - 1) = (0)$

*(ii)* $N(\alpha - 1) \perp R(\alpha - 1)$

*(iii)* $H = N(\alpha - 1) \oplus M$, *where* $M = \overline{R(\alpha - 1)}$.

**Proof.** The proof of (i) follows from [47, Proposition 2.1 (ii)].

To prove (ii), let $x \in N(\alpha - 1)$ and $y \in R(\alpha - 1)$. Then $\alpha(x) = x$ and $y = \alpha(z) - z$ for some $z \in H$. Therefore, $\alpha^*(x) = x$ and hence

$$\begin{aligned} \langle x, y \rangle &= \langle x, \alpha(z) - z \rangle \\ &= \langle x, \alpha(z) \rangle - \langle x, z \rangle \\ &= \langle \alpha^*(x), z \rangle - \langle x, z \rangle \\ &= \langle x, z \rangle - \langle x, z \rangle \\ &= 0. \end{aligned}$$

Thus $N(\alpha - 1) \perp R(\alpha - 1)$.

To prove (iii), it is enough to show that $N(\alpha - 1) = M^{\perp}$. By (ii) and the continuity of $\alpha$, $N(\alpha - 1) \perp M$. So, $N(\alpha - 1) \subseteq M^{\perp}$. Conversely, assume that $z \in M^{\perp}$. Then $\langle z, y \rangle = 0$ for all $y \in M$; in particular, $\langle z, (\alpha - 1)x \rangle = 0$ for all $x \in H$ because $R(\alpha - 1) \subseteq M$. Thus $\langle z, \alpha(x) \rangle = \langle z, x \rangle$ for all $x \in H$. So, $\langle \alpha^*(z), x \rangle = \langle z, x \rangle$ for all $x \in H$. This shows that $\langle \alpha^*(z) - z, x \rangle = 0$ for all $x \in H$. Therefore, $\alpha^*(z) - z = 0$ or $\alpha^*(z) = z$; that is, $z$ is a fixed point of $\alpha^*$ and hence by assumption $\alpha(z) = z$. That is, $z \in N(\alpha - 1)$. So, $M^{\perp} \subseteq N(\alpha - 1)$. Thus $N(\alpha - 1) = M^{\perp}$ and hence $H = N(\alpha - 1) \oplus M$. ■

**Proposition 4.2.2** *Let $\alpha$ be a bounded linear Banach operator on a Hilbert space $H$ such that $\alpha$ and $\alpha^*$ have common fixed points. Then there is a largest projection $P$ on $H$ such that $\alpha(Px) = P(x)$ for all $x \in H$.*

**Proof.** As in Proposition 4.2.1, put $M = \overline{R(\alpha - 1)}$. Let $f$ be the projection onto the closed subspace $M$ such that $f(H) = M$. Since $R(\alpha - 1)$ is $\alpha$-invariant, so is $M$. Also, $M^{\perp} = N(\alpha - 1)$ is $\alpha$-invariant. Thus $M$ reduces $\alpha$ and hence $\alpha$ commutes with $f$ ([18]). By Proposition 4.2.1, $(1-f)(R(\alpha-1)) =$

0; that is, $(1-f)(\alpha(x)-x)=0$ for all $x \in H$. If we put $P=(1-f)$, then $\alpha$ commutes with the projection $P$ and hence by the above equality, $P(\alpha(x)-x)=P(\alpha(x))-P(x)=\alpha(P(x))-P(x)=0$; that is, $P(\alpha(x))=P(x)$ for all $x \in H$. That $P$ is the largest projection follows from the orthogonality relations in Proposition 4.2.1 (see also [18]). ■

We conclude this chapter that with a result about an operator equation similar to the one considered in [47].

**Proposition 4.2.3** *Let $\alpha, \beta$ be invertible bounded linear multiplicative Banach operators on a normed algebra $X$ with identity such that $\alpha(x)+c\alpha^{-1}(x)=\beta(x)+c\beta^{-1}(x)$ for all $x \in X$, where $c$ is a real or complex number such that $|c| \geq 1$, $\|\alpha\|^2 \leq \frac{|c|}{2}, \|\beta\|^2 \leq \frac{|c|}{2}$ and if $\beta$ is inner, then $\alpha=\beta$.*

**Proof.** It follows from [47, Proposition 3.2] that $\alpha$ and $\beta$ commute. Therefore,

$$\begin{aligned}
(\alpha\beta - c)(\beta^{-1}-\alpha^{-1})(x) &= \alpha(x)-\alpha\beta\alpha^{-1}(x)-c\beta^{-1}(x)+c\alpha^{-1}(x)\\
&= \alpha(x)-\beta\alpha(\alpha^{-1}(x))-c\beta^{-1}(x)+c\alpha^{-1}(x)\\
&= \alpha(x)-\beta(x)-c\beta^{-1}(x)+c\alpha^{-1}(x)\\
&= (\alpha(x)+c\alpha^{-1}(x))-(\beta(x)+c\beta^{-1}(x))=0.
\end{aligned}$$

Put $(\beta^{-1}-\alpha^{-1})(x)=y$. Then we obtain $(\alpha\beta-c)(y)=0$; that is, $\alpha\beta(y)=cy$. Therefore, by assumption we get $|c|\,\|y\|=\|cy\|=\|\alpha\beta(y)\| \leq \|\alpha\|\,\|\beta\|\,\|y\| \leq \frac{|c|}{2}\|y\|$. That is, $|c|\,\|y\| \leq \frac{|c|}{2}\|y\|$. This implies that $\|y\|=0$ and hence $(\beta^{-1}-\alpha^{-1})(x)=0$ for all $x \in X$. That is, $\beta^{-1}(x)=\alpha^{-1}(x)$ for all $x \in X$. Since $\alpha$ is onto, therefore replacing $x$ by $\alpha(x)$, we get $\beta^{-1}(\alpha(x))=x$ or $\alpha(x)=\beta(x)$ for all $x \in X$. ■

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