Substitution Operators between Measurable Function Spaces

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Abstract

In this paper we will consider the substitution (weighted composition operators) on measurable function spaces and Fredholmness of these type operators will be investigated.

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1 Preliminaries and notations

In the next section we investigate a necessary and sufficient condition for a weighted composition operator $W = uC_\phi$ to be Fredholm. Fredholm weighted composition operators have been studied by H. Takagi [7] in the $L^p(\Sigma)$ setting. By using some properties of conditional expectation operator we omit the continuity hypothesis of $M_u$. In other words, we do not require that $u \in L^\infty(\Sigma)$. This is stated as a hypothesis in [7].

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space. By $L(X)$, we denote the linear space of all $\Sigma$-measurable functions on $X$. When we consider any sub-$\sigma$-algebra $A$ of $\Sigma$, we assume they are completed; i.e., $\mu(A) = 0$ implies $B \in A$ for any $B \subset A$. For any $\sigma$-finite algebra $A \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the $L^p$-space $L^p(X, A, \mu|_A)$ to $L^p(A)$, and denote its norm by $\|\|_p$. We define the support of a measurable function $f$ as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(A)$ as a subspace of $L^p(\Sigma)$ and as a Banach space. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. An atom of the measure $\mu$ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. It is easy to see that every $A$-measurable function $f \in L(X)$ is constant $\mu$-almost everywhere on an atom $A$. So for each $f \in L(X)$
and each atom $A$ we have $\int_A f \, d\mu = f(A)\mu(A)$. A measure with no atoms is called non-atomic.

Associated with each $\sigma$-algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot \mid \mathcal{A}) = E^\mathcal{A}(\cdot)$ on the set of all non-negative measurable functions $f$ or on the set of all functions $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, that is uniquely determined by the conditions

(i) $E^\mathcal{A}(f)$ is $\mathcal{A}$-measurable, and

(ii) if $A$ is any $\mathcal{A}$-measurable set for which $\int_A f \, d\mu$ exists, we have $\int_A f \, d\mu = \int_A E^\mathcal{A}(f) \, d\mu$.

The operator $E^\mathcal{A}$ is called conditional expectation operator. This operator is at the central idea of our work, and we list here some of its useful properties:

E1. $E^\mathcal{A}(f \circ T) = E^\mathcal{A}(f)(g \circ T)$.

E2. $E^\mathcal{A}(1) = 1$.

E3. $|E^\mathcal{A}(fg)|^2 \leq E^\mathcal{A}(|f|^2)E^\mathcal{A}(|g|^2)$.

E4. If $f > 0$, then $E^\mathcal{A}(f) > 0$.

Properties E1 and E2 imply that $E^\mathcal{A}(\cdot)$ is idempotent and $E^\mathcal{A}(L^p(\Sigma)) = L^p(\mathcal{A})$. So when $\mathcal{A} = \Sigma$, we have $E^\Sigma = I$ where $I$ is identity operator. Suppose that $\varphi$ is a mapping from $X$ into $X$ which is measurable, (i.e., $\varphi^{-1}(\Sigma) \subseteq \Sigma$) and $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to $\mu$ ($\mu \circ \varphi^{-1} \ll \mu$). Let $h$ be the Radon-Nikodym derivative, $h = \frac{d\mu \circ \varphi^{-1}}{d\mu}$. If we put $\mathcal{A} = \varphi^{-1}(\Sigma)$, it is easy to show that for each non-negative $\Sigma$-measurable function $f$ or for each $f \in L^p(\Sigma)$ ($p \geq 1$), there exists a $\Sigma$-measurable function $g$ such that $E^\varphi^{-1}(\Sigma)(f) = g \circ \varphi$. We can assume that the support of $g$ lies in the support of $h$, and there exists only one $g$ with this property. We then write $g = E^\varphi^{-1}(\Sigma)(f) \circ \varphi^{-1}$, though we make no assumption regarding the invertibility of $\varphi$ (see [1]). For a deeper study of the properties of $E$ see the paper [5].

Take a function $u$ in $L(X)$ and let $\varphi : X \to X$ be a non-singular measurable transformation; i.e., $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then the pair $(u, \varphi)$ induces a linear operator $uC_\varphi$ from $L^p(\Sigma)$ into $L(X)$ defined by

$$uC_\varphi(f) = u.f \circ \varphi \quad (f \in L^p(\Sigma)).$$

Here, the non-singularity of $\varphi$ guarantees that $uC_\varphi$ as a mapping of equivalence classes of functions on support $u$ is well defined. If $uC_\varphi$ takes $L^p(\Sigma)$ into $L^q(\Sigma)$ or $uC_\varphi$ is equivalently bounded, then we say that $uC_\varphi$ is a weighted composition operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ ($1 \leq p, q \leq \infty$). When $u \equiv 1$, we just have the composition operator $C_\varphi$ defined by $C_\varphi(f) = f \circ \varphi$. For more details see [6].
2 Fredholm weighted composition operators on $L^p$-spaces

Let $1 \leq p < \infty$, $1 \leq q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then it is a well known fact that each $g^* \in L^q(\Sigma)$ defines a bounded linear functional $F_{g^*}$ on $L^p(\Sigma)$ by

$$F_{g^*}(f) = \int fg^* \, d\mu \quad (f \in L^p(\Sigma)).$$

Moreover, the mapping $g^* \to F_{g^*}$ is an isometry from $L^q(\Sigma)$ onto $(L^p)^*(\Sigma)$, so the norm dual of $L^p(\Sigma)$ can be identified with $L^q(\Sigma)$. In the following theorem we compute the adjoint of $uC_\varphi$.

**Proposition 1** Let $W = uC_\varphi$ be a weighted composition operator on $L^p(\Sigma)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $W^*g^* = hE(u.g^*) \circ \varphi^{-1}$ for all $g^* \in L^q(\Sigma)$.

**Proof.** Take $A \in \Sigma$ such that $0 < \mu(A) < \infty$. For $g^* \in L^q(\Sigma)$ consider a bounded linear functional $F_{g^*}$ on $L^p(\Sigma)$ as above. Then we have

$$(W^*F_{g^*})(\chi_A) = F_{g^*}(W\chi_A) = \int (W\chi_A)g^* \, d\mu$$

$$= \int u.\chi_A \circ \varphi \, g^* \, d\mu = \int hE(u.g^*) \circ \varphi^{-1}\chi_A \, d\mu = F_{hE(u.g^*)\circ \varphi^{-1}\chi_A}.$$

Hence, $W^*F_{g^*} = F_{hE(u.g^*)\circ \varphi^{-1}}$. After identifying $(L^p)^*(\Sigma)$ with $L^q(\Sigma)$ and $g^*$ with $F_{g^*}$, we can write $W^*g^* = hE(u.g^*) \circ \varphi^{-1}$ for all $g^* \in L^q(\Sigma)$. \hfill $\square$

In the following theorem we investigate a necessary and sufficient condition for a weighted composition operator $W = uC_\varphi$ to be Fredholm. The proof of the theorem follows a similar method of proof as was used to prove Theorem 4.2 in [4] which is similar to a theorem of Takagi [7]. We use the symbols $\mathcal{N}(W)$ and $\mathcal{R}(W)$ to denote the kernel and the range of $W$, respectively. We recall that $W$ is said to be a Fredholm operator if $\mathcal{R}(W)$ is closed and if $\dim \mathcal{N}(W) < \infty$ and $\dim \mathcal{R}(W) < \infty$.

**Theorem 2** Suppose that $\mu$ is a non-atomic measure. Let $W = uC_\varphi$ be a weighted composition operator on $L^p(\Sigma)$. Then $W$ is a Fredholm operator if and only if $J = hE^{\varphi^{-1}}(\Sigma)(|u|^p) \circ \varphi^{-1} \geq \delta$ almost everywhere on $X$ for some $\delta > 0$.

**Proof.** Suppose that $W$ is a Fredholm operator. We first claim that $W$ is onto and takes an $f_0 \in L^p(\Sigma) \setminus \mathcal{R}(W)$. Since $\mathcal{R}(W)$ is closed, we can find a functional $L_{g^*}$ on $L^p(\Sigma)$ corresponding to $g^* \in L^q(\Sigma)$ ($\frac{1}{p} + \frac{1}{q} = 1$) which is defined as

$$L_{g^*}(f) = \int_X fg^* \, d\mu \quad \text{such that} \quad L_{g^*}(f_0) = 1 \quad \text{and} \quad L_{g^*}(\mathcal{R}(W)) = 0.$$

(1)
Hence the set $E_\delta = \{x \in X : \text{Re}(f_0 g^*)(x) \geq \delta\}$ must have positive measure for some $\delta > 0$. Since $\mu$ is non-atomic we can choose a sequence $\{E_n\}$ of subsets of $E_\delta$ with $0 < \mu(E_n) < \mu(E_\delta)$ and $E_n \cap E_m = \emptyset$ for $n \neq m$. Let $g^*_n = \chi_{E_n} g^*$. Then $g^*_n \in L^q(\Sigma)$ and is nonzero because

$$\text{Re} \int_X f_0 g^*_n \, d\mu \geq \delta \mu(E_n) > 0.$$ 

Evidently for any $f \in L^p(\Sigma)$, $\chi_{E_n} f$ is in $L^p(\Sigma)$, and so the right equality of (1) yields

$$\int_X f(W^* g^*_n) \, d\mu = \int_X f h E(ug^*_n) \circ \varphi^{-1} \, d\mu = \int_{E_n} f E(ug^*) \circ \varphi^{-1} \, d\mu \circ \varphi^{-1}$$

$$= \int_{\varphi^{-1}(E_n)} f \circ \varphi E(ug^*) \, d\mu = \int_{\varphi^{-1}(E_n)} ug^* f \circ \varphi \, d\mu = \int_X g^* u f \circ \varphi(\chi_{E_n} \circ \varphi) \, d\mu$$

$$= \int_X g^* u (f \chi_{E_n}) \circ \varphi \, d\mu = \int_X g^* W(f \chi_{E_n}) \, d\mu = 0.$$ 

This implies that $g^*_n \in \mathcal{N}(W^*)$. Thus the sequence $\{g^*_n\}$ forms a linearly independent subset of $\mathcal{N}(W^*)$. This contradicts the fact that $\dim \mathcal{N}(W^*) = \text{codim } \mathcal{R}(W) < \infty$. Hence $W$ is onto. Next we put $Z(J) = \{x : J(x) = 0\}$. Now we claim that $\mu(Z(J)) = 0$. For, if $\mu(Z(J)) > 0$, there exists a subset $F$ of $Z(J)$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(W)$, then there exists $f \in L^p(\Sigma)$ such that $\chi_F = W f$. Then

$$\mu(F) = \int_F |W f|^p \, d\mu \int_F |f|^p \, d\mu = 0$$

and this is a contradiction. So $\chi_F \in L^p(\Sigma) \setminus \mathcal{R}(W)$, which contradicts the fact that $W$ is onto. Also since $\mu(Z(J)) = 0$ and $\mu \circ \varphi^{-1} \ll \mu$ we have $\mu(Z(J \circ \varphi)) = 0$. For each $n = 1, 2, \ldots$ let

$$H_n = \left\{ x \in X : \frac{\|J \circ \varphi\|\infty}{(n + 1)^2} < J \circ \varphi(x) \leq \frac{\|J \circ \varphi\|\infty}{n^2} \right\},$$

and $H = \{n : \mu(H_n) > 0\}$. Then the $H_n$'s are pairwise disjoint and $X = \bigcup_{n=1}^{\infty} H_n$.

Define

$$f(x) = \begin{cases} \left(\frac{J \circ \varphi(x)}{\mu(H_n)}\right)^{\frac{1}{p}} & \text{if } x \in H_n, n \in H, \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$\int_X |f|^p \, d\mu = \sum_{n \in H} \int_{H_n} \frac{J \circ \varphi(x)}{\mu(H_n)} \, d\mu \leq \sum_{n \in H} \frac{\|J \circ \varphi\|\infty}{n^2} \leq \|J \circ \varphi\|\infty \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$
so \( f \in L^p(\Sigma) \). If \( g \in L^p(\Sigma) \) is such that \( Wg = f \), then

\[
\int_X E^{\phi^{-1}}(|u|^p)|g|^p \circ \phi \, d\mu = \int_X E^{\phi^{-1}}(|f|^p) \, d\mu.
\]

It follows that

\[
\int_X hE^{\phi^{-1}}(|u|^p) \circ \phi^{-1}|g|^p \, d\mu = \int_X hE^{\phi^{-1}}(|f|^p) \circ \phi^{-1} \, d\mu.
\]

Thus \(|g|^p = \frac{hE^{\phi^{-1}}(|f|^p) \circ \phi^{-1}}{\int_X hE^{\phi^{-1}}(|u|^p) \circ \phi^{-1}}\) on \( Z(J) \). Since \( \mu(Z(J)) = 0 \), it follows that

\[
\int_X |g|^p \, d\mu = \int_X E^{\phi^{-1}}(|f|^p) \circ \phi^{-1} \, d\mu = 1.
\]

This implies that \( H \) must be a finite set. Thus there is an \( n_0 \) such that \( n \geq n_0 \) implies \( \mu(H_n) = 0 \) and so

\[
\mu \left( \left\{ x \in X : J \circ \phi(x) \leq \frac{\|J \circ \phi\|_\infty}{n_0} \right\} \right) = \mu \left( \bigcup_{n=n_0}^{\infty} H_n \cup Z(J \circ \phi) \right) = 0.
\]

Therefore we obtain \( J \circ \phi \geq \frac{\|J \circ \phi\|_\infty}{n_0} \) almost everywhere on \( X \). Since \( \mathcal{N}(W) = L^p(Z(J)) = 0 \), \( \mu(Z(J)) = 0 \) and \( \dim \mathcal{N}(W) = 0 \) and then \( \phi \) is essentially surjective. Hence \( J \geq \frac{\|J \circ \phi\|_\infty}{n_0} (= \delta) \) almost everywhere on \( X \).

Conversely, suppose that \( J \geq \delta \) almost everywhere for some \( \delta > 0 \). Since \( h > 0 \) and for each \( f \in L^p(\Sigma) \), \( \|Wf\|_p = (\int_X |f|^p \, d\mu)^{1/p} \geq \delta^{1/p} \|f\|_p \), it follows that \( W \) and \( C_\phi \) are injective and \( \mathcal{R}(W) \) is closed. Also since \( W = M_u C_\phi \), we deduce that \( M_u \) is injective and so \( \mu(Z(u)) = 0 \). Now let \( g^* \in \mathcal{N}(W^*) \). Then \( W^* g^* = hE^{\phi^{-1}}(ug^*) \circ \phi^{-1} = 0 \) and so \( E^{\phi^{-1}}(ug^*) \circ \phi^{-1} = 0 \). It follows that \( g^* = 0 \). Thus \( \text{codim} \mathcal{R}(W) = \dim \mathcal{N}(W^*) = 0 \). Therefore the theorem is proved. \( \square \)

**Corollary 3** Suppose \( M_u \) and \( C_\phi \) are both bounded linear operators on \( L^p(\Sigma) \) and \( \mu \) is a non-atomic measure. Then

(i) \( M_u \) is Fredholm if and only if \( |u| \geq \delta \) on \( X \) for some \( \delta > 0 \).

(ii) \( C_\phi \) is Fredholm if and only if \( h \geq \delta \) on \( X \) for some \( \delta > 0 \).
Remark 4 One of the interesting features of a weighted composition operator is that the multiplication operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let $X$ be $(0, 1)$, $\Sigma$ be the Borel sets, and $\mu$ be the Lebesgue measure. Let $\varphi$ be the map $\varphi(x) = 3\sqrt{x}$ and $u(x) = 1/\sqrt{x}$ on $(0, 1)$. Then $M_u$ does not define a bounded operator from $L^1(\Sigma)$ into $L^1(\Sigma)$. However a simple computation shows that $J(x) = 3\sqrt{x} \in L^\infty(\Sigma)$ and so $Wf(x) = 1/\sqrt{x}f(3\sqrt{x})$ is bounded operator on $L^1(\Sigma)$.

References