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Multiplicity Results for an Elliptic System

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Abstract

In this paper, we will be concerned with the existence of solutions and their multiplicities for an elliptic system modelling two subpopulations competing for resources.

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1 Introduction

In [1], the authors studied the following elliptic system:

$$\begin{cases} -\Delta u = \sigma(x,u)\,v - e(x)\,u - c(x)\,u\,(u+v) & \text{in } \Omega, \\ -\Delta v = b(x,v)\,u - f(x)\,v - d(x)\,v\,(u+v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The system (1.1) is modelling two subpopulations of the same species competing for resources. The function u represents the concentration of the adult population and v the concentration of the young population. The two populations live in the domain Ω which is supposed bounded and regular in $\mathbb{R}^n$.

It is proved in [1] under suitable conditions that system (1.1) has a unique positive solution. Using Lyapounov-Schmidt reduction method (see [2]), we will show that problem (1.1) can have more than one solution in some situations.

Let λ_1 be the first eigenvalue of the operator $(-\Delta + e)$ with homogeneous boundary conditions.

Suppose that :

$$\begin{aligned}\sigma(x,u) &= \lambda_1+\varepsilon\sigma_1(x,u),\\ b(x,v) &= \lambda_1+\varepsilon b_1(x,v),\\ c(x) &= \varepsilon\, c_1(x),\\ d(x) &= \varepsilon\, c_2(x),\\ f(x) &= e(x)+\varepsilon\, f_1(x),\end{aligned}$$

where σ_1, b_1, c_1 and c_2, f_1 are bounded functions and ε is small enough.

Let $X := W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and $Y = L^p(\Omega)$ with $p > 1$, and define the operators H and B respectively by

$$H:\ X\times X\to Y\times Y,$$
$$H(u,v) := \begin{pmatrix}\Delta u - eu+\lambda_1 v\\ \Delta v - ev+\lambda_1 u\end{pmatrix}$$

and

$$B:\ X\times X\to Y\times Y,$$
$$B(u,v) := \begin{pmatrix}\sigma_1 v - c_1 u(u+v)\\ b_1 u - c_2 v(u+v) - f_1 u\end{pmatrix}.$$

Hence, problem (1.1) is equivalent to

$$H(u,v)+\varepsilon\, B(u,v) = 0. \tag{1.2}$$

2 Main Results

(For more details see [3].)

i/ Let φ_1 be the eigenfunction associated to λ_1. Then

$$\operatorname{Ker} H = \left\{(u,v)\in X^2 |\ (u,v) = s(\varphi_1,\varphi_1), \qquad s\in\mathbb{R}\right\}.$$

ii/ Denote by X_1 and Y_1 respectively the complementary subspaces of $\operatorname{Ker} H$ in X and Y respectively, *i.e.*,

$$\begin{aligned} X &= \operatorname{Ker} H\oplus X_1,\\ Y &= \operatorname{Ker} H\oplus Y_1,\end{aligned}$$

and let P and Q be respectively the orthogonal projections on X_1 and Y_1.

Proposition 1 *The restriction of the operator QH to X_1 is an invertible operator.*

Applying Q and $(I-Q)$ to (1.2) and taking into account Proposition 1, we will see that (1.2) is equivalent to

$$F(s,\varepsilon) := (I-Q)B\left[s(\varphi_1,\varphi_1)+U(s,\varepsilon)\right]=0, \tag{2.1}$$

where $U(s,\varepsilon)$ is a solution of the following fixed point problem:

$$U=-\varepsilon(QH)^{-1}QB\left[(s,s)\varphi_1+U\right] \quad \text{and} \quad U=P(u,v).$$

Theorem 1 *Suppose that:*

i/ $\alpha=\int_\Omega(\sigma_1+b_1-f_1)\varphi_1^2\,dx\neq 0$*,*

ii/ $\beta=\int_\Omega(c_1+c_2)\varphi_1^3\,dx\neq 0$*.*

Then problem (2.1) *has two solutions of the form:*

$$\begin{aligned} &\xi_1(s,\varepsilon)=S_0(\varepsilon)(\varphi_{1,}\varphi_1)+U(S_0(\varepsilon),\varepsilon)\\ \textit{with}\quad &S_0:\ (-\varepsilon,\varepsilon)\to V_0 \text{ — \textit{a neighbourhood of} } s=0\end{aligned}$$

and

$$\begin{aligned} &\xi_2(s,\varepsilon)=S_1(\varepsilon)(\varphi_{1,}\varphi_1)+U(S_1(\varepsilon),\varepsilon)\\ \textit{with}\quad &S_1:\ (-\varepsilon,\varepsilon)\to V_* \text{ — \textit{a neighbourhood of} } s=s^*=\frac{\alpha}{2\beta}\ .\end{aligned}$$

Proof. It is easy to see that

$$(I-Q)\begin{pmatrix} u\\ v\end{pmatrix}=\frac{\varphi_1}{2}\begin{pmatrix}\int_\Omega(u+v)\varphi_1\,dx\\ \int_\Omega(u+v)\varphi_1\,dx\end{pmatrix}.$$

Hence for $\varepsilon=0$, equation (2.1) becomes

$$F(s,0)=(I-Q)B\left[s\left(\varphi_1,\varphi_1\right)\right]=0,$$

which implies that

$$(I-Q)\begin{pmatrix}\sigma_1 s\varphi_1-2c_1s^2\varphi_1^2\\ b_1s\varphi_1-f_1s\varphi_1-2c_2s^2\varphi_1^2\end{pmatrix}=0$$

or equivalentely

$$F(s,0)=0 \qquad \Longleftrightarrow \qquad F(s):=\alpha s-2s^2\beta=0,$$

where

$$\begin{aligned}\alpha &= \int_\Omega(\sigma_1+b_1-f_1)\varphi_1^2\,dx,\\ \beta &= \int_\Omega\left(c_1+c_2\right)\varphi_1^3\,dx.\end{aligned}$$

It suffices now to apply the implicit functions theorem to deduce the existence of at least two solutions. ∎

References

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