Overview

1. Validity and Propositional Equivalences

2. Showing Equivalence using Truth Tables

3. Showing Equivalence using Symbolic Manipulations

4. Preview: Predicates and Quantifiers

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Validity and Equivalences

- As pointed out in the preceding lecture, a given sentence can be represented using different logical expressions.

- If the different logical expressions representing a sentence are all correct, then they should have the same truth values.

Definition 1 *A proposition according to its possible truth value may be:*

- *A proposition $\mathcal{F}$ is* valid (tautology) *if $\mathcal{F}$ is true under every interpretation.*
- *A proposition $\mathcal{F}$ is* contradiction *if $\mathcal{F}$ is false under every interpretation.*
- *A proposition that is neither a tautology nor a contradiction is called a* contingency*.*
- *Two sentences $\mathcal{F}$ and $\mathcal{G}$ are* logically equivalent *if, under every interpretation, $\mathcal{F}$ has the same truth-value as $\mathcal{G}$.* ■

The most straightforward way to determine whether a sentence is valid or contradiction, is by using a *truth table*.

Example: *The sentence*

$$\mathcal{F} : \neg(p \vee q) \equiv ((\neg p) \wedge (\neg q)).$$

is valid.

The corresponding truth table is

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$	$\mathcal{F}$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

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Example: *On the other hand, the sentence*

$$\mathcal{G} : \ (p \rightarrow \ q) \ \rightarrow \ (\neg p \rightarrow \neg q)$$

is not valid.

The corresponding truth table is

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\mathcal{G}$
T	T	T	F	F	T	T
T	F	F	F	T	T	T
F	T	T	T	F	F	F
F	F	T	T	T	T	T

Example: *Is the implication shown below a tautology?*

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

Notice that the given logical expression consists of 3 variables p, q and r and thus the truth table will have $2^3 = 8$ rows of truth values.

Example: *Use a truth table to verify the equivalence*

$$\mathcal{H} : \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q.$$

p	q	$\neg p$	$\neg q$	$p \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

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Some Famous Logical Equivalences

Sometime it is useful to replace a logical statement with another logical statement having the same truth value during symbolic manipulation of logical expressions. The following are some popular identities needed for manipulating logical expressions.

Equivalence	**Name**
$P \wedge T \equiv P$ $P \vee F \equiv P$	Identity laws
$P \vee T \equiv T$ $P \wedge F \equiv F$	Domination laws
$P \vee P \equiv P$ $P \wedge P \equiv P$	Idempotent laws
$\neg(\neg P) \equiv P$	Double negation law
$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative laws
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative laws
$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	Distributive laws
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	DeMorgan's laws
$P \rightarrow Q \equiv \neg P \vee Q$	Implication
$P \vee \neg P \equiv T$	Tautology
$P \wedge \neg P \equiv F$	Contradiction
$(P \rightarrow Q) \wedge (Q \rightarrow P) \equiv P \leftrightarrow Q$	Equivalence
$((P \leftrightarrow Q) \leftrightarrow R) \equiv (P \leftrightarrow (Q \leftrightarrow R))$	Associativity of $\leftrightarrow$
$P \vee (P \wedge Q) \equiv P$ $P \wedge (P \vee Q) \equiv P$	Absorption laws

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Example: *Use truth tables to verify the logical identities in the table above.*

Example: *Is the implication shown below a tautology?*

$$(\neg P \vee Q) \vee (P \wedge \neg Q)$$

Solution:

Transformation	**Remark**
$(\neg P \vee Q) \vee (P \wedge \neg Q)$	Given
$((\neg P \vee Q) \vee P) \wedge ((\neg P \vee Q) \vee \neg Q)$	Distributive law
$((\neg P \vee P) \vee Q) \wedge (\neg P \vee (Q \vee \neg Q))$	Assoc. and comm. laws
$(T \vee P) \vee Q) \wedge (\neg P \vee T)$	$(P \vee \neg P) \equiv T$
$T \wedge T$	$(T \vee P) \equiv T$
T	Proof complete.

Note: We can, alternatively, prove the given expression in two steps by using the DeMorgan's law followed by the law $p \vee \neg p \equiv T$, thus:

Transformation	**Remark**
$(\neg P \vee Q) \vee (P \wedge \neg Q)$	Given
$\neg(P \wedge \neg Q) \vee (P \wedge \neg Q)$	DeMorgan law on $(\neg P \vee Q)$
T	$(A \vee \neg A) \equiv T$

Example: *Is the implication shown below a tautology?*

$$[(P \rightarrow Q) \wedge (P \rightarrow R)] \rightarrow [P \rightarrow (Q \vee R)]$$

Solution:

Transformation	**Remark**
$\neg[(\neg P \vee Q) \wedge (\neg P \vee R)] \vee [\neg P \vee (Q \vee R)]$	`Definition of` $\rightarrow$
$[\neg(\neg P \vee Q) \vee \neg(\neg P \vee R)] \vee [\neg P \vee (Q \vee R)]$	`DeMorgans' law.`
$[\neg(\neg P \vee Q) \vee (\neg P \vee Q)] \vee \neg(\neg P \vee R) \vee R$	`Associativity.`
$T \vee \neg(\neg P \vee R) \vee R$	$(P \vee \neg P) \equiv T.$
T	$(T \vee P) \equiv T$

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