3

Rational Election Procedures

Author: Russell B. Campbell, Department of Mathematics and Computer Science, The University of Northern Iowa.

Prerequisites: The prerequisites for this chapter are sets and relations. See Sections 2.1 and 2.2 and Chapter 8 of *Discrete Mathematics and Its Applications*.

Introduction

The word democracy, derived from the Greek *dēmos* (the people) and *kratia* (power), means rule or authority of the people. If a democracy is to be successful, it must manifest the will of the people. A collective will reflecting the diverse wills of the individuals must be followed. Over the centuries several means to achieve this end have been proposed. In 1951 Kenneth Arrow* showed that no such collective will exists. Hence there is inherent inequity which cannot be avoided.

* Kenneth J. Arrow (1921–) published *Social Choice and Individual Values* [1] in 1951, which demonstrated that no such collective will exists. That monograph was essentially his Ph.D. dissertation, and is the primary reason that he received the Nobel prize in Economics in 1972. Arrow has been on the faculty of Stanford University for forty years during which time he has received numerous fellowships and honorary degrees.

Individuals are often unhappy if their candidate does not win an election. But sometimes the outcome indeed does not reflect what the majority of the people want. One example is that if 60% of the voters are liberal, but divide their votes evenly between two candidates, the single conservative candidate wins with only 40% of the vote. Another example occurs in an election with a runoff, where a candidate who would have won any two-way race may be eliminated from the runoff. Elections should not result in outcomes the voters did not want.

A *rational election procedure* is one under which each individual's vote will positively affect his candidate's standing in the outcome. (This notion is made precise below.) There is no rational election procedure if there are more than two candidates. This chapter illustrates some of the reasons why. But more importantly, it illustrates how the concept of relations and various properties of relations can be used to concretely formulate the vague notion of fairness in election procedures. The election procedures in the latter half of this chapter employ some of the notation introduced in the first half, but do not require understanding the theorems.

The Problem

The problem was originally posed as choosing a *social welfare function* (i.e., a function which converts the preferences of individuals for alternative social states into a single collective preference schedule for alternative social states). We shall specialize to the problem of conducting an election and refer to the social welfare function which Arrow discussed as a rational election procedure (REP). Although elections are usually discussed in the context of electing government officials, ranking college football teams by a vote of sportswriters also requires an election procedure (EP) to convert the individual rankings into a collective ranking. In order to define an election procedure, and add properties which will make it a rational election procedure, it is necessary to introduce some notation.

Let n denote the number of alternative choices (for example, candidates A, B, C, $\ldots$). The fundamental assumption is that each voter has preferences among the candidates:

> For every voter i, there is a relation R_i which represents his preferences for the candidates. The statement AR_iB means that voter i likes candidate A at least as well as candidate B. Each relation R_i must satisfy two axioms: (i) transitivity and (ii) connectivity.

Transitivity provides order to the preferences: if a voter prefers A to D and also prefers D to B, it is reasonable to expect that the voter prefer A to B. *Connectivity* requires that the voter will have a preference (or indifference) between every pair of candidates; it means that if presented with two alternative candidates, a voter will either prefer one to the other or be indifferent between them (i.e., consider them a tie), thus every pair of candidates is comparable. A formal definition of connected for a relation R_i follows.

Definition 1 A relation R_i is *connected* if, given any two alternatives X and Y, either XR_iY or YR_iX (both may hold). □

Together transitivity and connectivity require that every individual must be able to rank the candidates preferentially, but allows for indifference among some of the candidates.

Definition 2 A relation which is connected and transitive is called a *weak order.* □

Example 1 A weak order may be represented in the following manner:

$$\{C \succ B, A \succ E \succ D, F\},$$

where the candidates to the left of $\succ$ are preferred to those to the right, and the voter is indifferent between candidates separated by commas. Thus, in this example, the voter prefers candidate C to all alternatives; is indifferent between B and A which are preferred to D, E, and F; and is also indifferent between D and F. Every possible preference of a voter may be represented in the above manner. □

Definition 3 Let W be the set of all weak orders on n candidates and assume there are N voters. An *election procedure (EP)* is a function $W^N \rightarrow W$, i.e., a mapping from sets of individual preferences to collective preferences. □

Transitivity is a property which was used to define both an equivalence relation and a partial order relation. Connectivity forces reflexivity (see Exercise 3), which was also part of the definition of both an equivalence relation and a partial order relation. Therefore, if a weak order is symmetric, it is also an equivalence relation. The characterization of relations which are both weak orders and equivalence relations is determined by connectivity and symmetry.

Theorem 1 If a relation R is connected and symmetric, it is the universal relation (everything is related to everything, i.e., ARB for all A and B).

Proof: We prove that $A \not R B$ cannot happen. Suppose $A \not R B$. By connectivity BRA, and symmetry forces ARB, which is a contradiction. Therefore ARB.■

The universal relation is transitive. Hence the only equivalence relation (which by definition is symmetric) which is a weak order (which by definition is connected) is the universal relation, which has a single equivalence class. The only weak order on a the set of six candidates which is an equivalence relation can be represented as

$$\{A, B, C, D, E, F\},$$

i.e., indifference among all candidates.

A weak order is by definition transitive, and connected implies reflexive. Hence if a weak order is antisymmetric, it is also a partial order. Indifference between distinct alternatives A and B entails that ARB and BRA, hence antisymmetry precludes indifference among distinct alternatives.

Definition 4 A relation which is antisymmetric, transitive, and connected is called a *total order* or *linear order* (see Section 8.6 of *Discrete Mathematics and Its Applications*). □

A weak order which is a partial order (hence a total order) on a set of six candidates can be represented as a single "chain", such as

$$\{A \succ B \succ E \succ C \succ D \succ F\}.$$

A weak order on a set with more than one element cannot be both an equivalence relation and a partial order relation (see Exercise 7). The weak order displayed in Example 1 is neither an equivalence relation nor a partial order.

An important feature of weak orders is that they only express preferences qualitatively. No numerical values are assigned to the preference rankings. We are not allowing voting procedures which assign 5 points to a first choice, 3 to a second, and 1 to a third choice, and then add up the total points to determine the winner. Such procedures are used for scoring athletic contests such as track and field, and ranking sports teams based on the votes of sportswriters; but Arrow did not allow such assignments because he felt it is not possible to quantify preferences, especially on a scale which is consistent between voters.

Desired Properties of the REP

Arrow summarized the heuristic notions of fair and rational with two properties which the collective relation R resulting from the EP must have in order to justify characterizing an EP as rational:

(i) It must positively reflect the wills of individuals (PR).
(ii) It must be independent of irrelevant alternatives (IA).

To formulate these conditions precisely, we introduce a relation derived from a weak order.

Definition 5 A *strict preference* is a relation P defined by

$$XPY \Leftrightarrow XRY \wedge Y\not{R}X,$$

where R is a weak order. □

This provides that X is related to Y in P if and only if X is strictly preferred to Y in R. P is irreflexive, and hence not connected. We use accents (such as $\widehat{\ }$, $'$, $\widetilde{\ }$) to indicate the correspondence of various relations (relations which are derived from each other share the same accent): $\widetilde{R}$ is the collective preference which comes from the individual preferences $\{\widetilde{R}_i\}$; $\widehat{P} = \widehat{R} - \widehat{R}^{-1}$ (the strict preference derived from the weak order $\widehat{R}$). Relations without any accent also correspond to each other.

Positively reflecting individual preferences is the heuristic notion that if a voter changes his vote in favor of a candidate, that candidate should not fare worse in the outcome of the election. Property PR is defined by contrasting two sets of individual preferences $\{R_i\}$ and $\{\widetilde{R}_i\}$ which are identical with respect to all candidates except Z, but some of the $\widetilde{R}_i$ may manifest a greater preference for Z than the corresponding R_i. Property PR then requires that $\widetilde{R}$ ranks Z at least as high as R does. A more concise statement of the definition follows:

Definition 6 An election procedure (EP) *positively reflects the wills of individuals* (satisfies property PR) if the following holds:

If, whenever two sets of individual preferences $\{R_i\}$ and $\{\widetilde{R}_i\}$ satisfy for all X and Y different from a specified Z and for all i:

1. $XR_iY \Leftrightarrow X\widetilde{R}_iY$,
2. $ZR_iX \Rightarrow Z\widetilde{R}_iX$, and
3. $ZP_iX \Rightarrow Z\widetilde{P}_iX$,

then $ZPX \Rightarrow Z\widetilde{P}X$. □

In other words, if, all other preferences being the same, there is a stronger preference for Z in the individual rankings, Z will not manifest a lower preference in the collective function (it could be the same).

Independence of irrelevant alternatives says that if a candidate withdraws from a race, it will not affect the relative rankings of the other candidates. This is expressed by considering relations on a set T restricted to a set $S \subset T$ (i.e., if R is a relation on T, R restricted to $S \subset T$ is $R \cap (S \times S)$).

Definition 7 *Independence of irrelevant alternatives* is satisfied if, whenever $R_i = \widetilde{R}_i$ for all i when the relations are restricted to S, then $R = \widetilde{R}$ when restricted to S. □

The preceding two definitions allow us to define when an election procedure is rational

Definition 8 A *rational election procedure* (REP) is an election procedure (EP) which satisfies properties PR and IA. □

Plurality voting is not a REP as illustrated in the Introduction with the example of two liberal versus one conservative candidate. If either liberal were to withdraw, the other would win; hence the victory of the conservative over either liberal is not independent of irrelevant alternatives. Sequential runoffs also violates IA as the example in the Introduction shows.

The General Possibility Theorem

The *general possibility theorem* of Arrow is really an *impossibility* theorem. It states that if there are at least two voters and three candidates, there is no rational election procedure which reflects the preferences of the voters. More specifically, it states that the only functions from the individual preferences $\{R_i\}$ (which are weak orders) to a collective preference R (which is a weak order) which satisfy positively reflecting individual preferences (PR) and independence of irrelevant alternatives (IA) either select one of the individual preferences (i.e., $R = R_i$ for some i) or impose a weak order between at least two of the candidates independent of the individual preferences (i.e., XRY for some X and Y independent of the R_i). The former provides a dictatorship by the ith voter, while the latter, which is referred to as **externally imposed**, does not reflect anyone's preferences. Neither of these election procedure alternatives is consistent with our notion of democracy.

To simplify the proof of the general possibility theorem, we restrict our attention to the case of only two voters and three candidates.

Theorem 2 Arrow's General Possibility Theorem The only functions $F : W^2 \to W$ which satisfy PR and IA are

1. $F(R_1, R_2) = R_1$ or $F(R_1, R_2) = R_2$ (i.e., dictatorships), or
2. $F(R_1, R_2) = R$ with XRY for some X and Y independent of R_1 and R_2 (i.e., externally imposed).

(R_1 and R_2 are weak orders on the three candidates; the subscripts identify voters 1 and 2 respectively.) ■

For notational convenience, we denote the three candidates with A, B, and C; and use X and Y generically for the candidates A, B, and C. The proof is based on a series of lemmas under the assumption that the REP is not externally imposed, which successively characterize the REP until it is shown to be a dictatorship. A contradiction is then obtained by applying the lemmas.

Lemma 1 If XP_1Y and XP_2Y, then XPY.

Recall that P is the strict preference defined from the weak order R which results from weak orders R_1 and R_2 which define the strict preferences P_1 and P_2; the mapping which produces R is assumed to satisfy properties IA and PR in accordance with the hypothesis of the theorem. (This is certainly a desirable property. If both individuals prefer X to Y, the collective preference should, but it must be shown.)

Proof: If YRX for all choices of P_1 and P_2, R is an externally imposed weak order, which is one of the forms of the function in the conclusion of the theorem, so assume $X\widehat{P}Y$ for $\widehat{P}_1$ and $\widehat{P}_2$ (i.e., such $\widehat{P}_i$ exist). We may alter the preference orderings $\widehat{P}_1$ and $\widehat{P}_2$ by making X strictly preferred to the other alternatives (if that is not already the case) while leaving the relationships between the other two alternatives unchanged; denote these preferences as P_1' and P_2'. Because the REP positively reflects individual preferences (PR), $XP'Y$. Hence, by property IA this will hold for all preferences satisfying the hypothesis of Lemma 1. ■

The next lemma treats a circumstance when the preferences of the two voters differ, which the following lemma will show cannot occur.

Lemma 2 If whenever XP_1Y and YP_2X, XPY; then whenever XP_1Y, XPY. ■

This is also intuitive, it says that if individual 1's preferences win out when opposed, they will also win when not opposed. The proof (see Exercise 8) follows from property PR.

Before proceeding, we need to define a class of relations which will be denoted with I. I stands for **indifferent**, i.e., $XIY \Leftrightarrow XRY \wedge YRX$. We leave it as an exercise to show that indifference is an equivalence relation (see Exercise 4). This notation allows us to characterize collective preference functions when voter preferences are opposite.

Lemma 3 If XP_1Y and YP_2X, then XIY.

This says that if the two voters' preferences for two candidates are opposite, the REP must show indifference between the two candidates ("their votes cancel").

Proof: The alternative is that the REP provides P satisfying XPY (or YPX which would be handled analogously). Property IA extends "XP_1Y and YP_2X and XPY" to "whenever XP_1Y and YP_2X, XPY", which is the hypothesis of Lemma 2. The conclusion of Lemma 2 will be used to draw a contradiction.

Assume that AP_1B, BP_2A, and APB. By property IA this holds for all sets of voter preferences $\{\dot{P}_i\}$ with voter 1 preferring A to B and voter 2 preferring B to A. Consider $\dot{P}_1$ prescribing the preference $A \succ B \succ C$ and $\dot{P}_2$ prescribing the preference $B \succ C \succ A$. (We use $X \succ Y$ to denote XPY, commas separate alternatives between which the voter is indifferent.) It follows by Lemma 1 that $B\dot{P}C$, hence $A\dot{P}C$ by transitivity for these preferences. As noted above, a single case ($\{\dot{P}_i\}$) satisfies the hypotheses of Lemma 2 by property IA, hence $AP_1C \Rightarrow APC$. We now have $AP_1B \Rightarrow APB$ and $AP_1C \Rightarrow APC$.

Next consider $\widehat{P}_1$ prescribing the preference $B \succ A \succ C$ and $\widehat{P}_2$ prescribing the preference $C \succ B \succ A$. It follows by Lemma 1 that $B\widehat{P}A$. $A\widehat{P}C$ since it is true for $\widehat{P}_1$. Hence $B\widehat{P}C$ by transitivity. $B\widehat{P}_1C$, $C\widehat{P}_2B$ and $B\widehat{P}C$ is equivalent to the hypothesis of Lemma 2. We now have $AP_1B \Rightarrow APB$, $AP_1C \Rightarrow APC$, and $BP_1C \Rightarrow BPC$.

Consideration of the individual preferences $B \succ C \succ A$ and $C \succ A \succ B$ adds $BP_1A \Rightarrow BPA$ to our list. Continuing in this fashion we can show that the preference of the first individual governs, i.e., the REP is given by $R = F(R_1, R_2) = R_1$ (a dictatorship). This proves Lemma 3 since the assumption $X\not{I}Y$ implied that the REP is a dictatorship, which is one of the forms of the REP specified in Theorem 2. Our initial choice AP_1B, BP_2A, and APB was made without loss of generality, hence provides the result for arbitrary X and Y (the reader may repeat the proof with BPA and all the other 2–permutations of A, B, and C). ■

These lemmas allow us to proceed with the proof of the general possibility theorem.

Proof of Theorem 2: Consider the preferences P_1 prescribing $A \succ B \succ C$ and P_2 prescribing $C \succ A \succ B$. It follows from Lemma 1 that APB and from Lemma 3 that AIC and BIC. The latter two relationships imply AIB by transitivity, which contradicts APB. Hence there is no REP with the desired properties. ■

Extension to more than three candidates is immediate since IA allows us to restrict any REP to three candidates, but extension to more than two voters is more recondite and will not be proven here. There is, however, an additional assumption (which is included in many definitions of "fair") that simplifies the proof. This assumption is symmetry, both with respect to voters and with respect to candidates.

Symmetric Election Procedures

The hypotheses of Theorem 2 put no restriction on the REP (rational election procedure) other than that it provide a weak order and satisfy properties PR and IA. It is reasonable to further require that the election procedure be symmetric with respect to both candidates and voters. *Symmetry* with respect to candidates means that no candidate has an inherent advantage built into the election procedure, i.e., all candidates are entering the race as equals. A formal characterization is that if XRY (or XPY or XIY) and everyone interchanged their votes for X and Y, then YRX (or YPX or YIX). *Symmetry* with respect to voters means that all voters are equal, it only matters what votes are cast, not who cast them. This is formally stated as: permuting the indices of the voters associated with the preference schedules, without changing the preference schedules, will leave the resultant R unchanged.

There are circumstances where these assumptions are not reasonable for constructing a REP, such as declaring the incumbent the winner if there is a tie or weighting the votes of women as half the votes of men, but these are rather anomalous.

Assuming symmetry greatly simplifies the proof of Lemma 3, which reduces to the conclusion that indifference is the only symmetric weak order (see Exercise 9). The following lemmas are preparation for the proof that in the symmetric case, if there are at least three candidates, there is no REP satisfying PR and IA other than total indifference for any number of voters.

Lemma 4 If XP_iY for all i, then XPY.

Proof: The proof of Lemma 4 is is analogous to the proof of Lemma 1. If there is not total indifference, there is a set of preference schedules under which some candidate is preferred to another. By symmetry there must be a set of preference schedules for each pair of candidates and, in particular, a set of preferences yielding XPY. By PR, we still have XPY if X is raised above Y on each schedule in that set, and Lemma 4 follows by IA. ■

The following lemma provides an easy way to determine the collective preference if none of the individual preference schedules contain indifference.

Lemma 5 If there is no indifference in the individual preference schedules and $|\{i|XP_iY\}| > |\{i|YP_iX\}|$, then XPY.

(Note that $|\ |$ denotes cardinality. This says that if more people strictly prefer X to Y than Y to X, then the collective will will strictly prefer X to Y.)

Proof: Construct individual preference schedules for three candidates of the form $X \succ Z \succ Y$ and $Z \succ Y \succ X$ where there are more of the former than the latter. By symmetry (and PR), XRY and XRZ. By Lemma 4, ZPY. If XIY, ZPX by transitivity which contradicts XRZ. Because IA extends this special case to the hypothesis of Lemma 5, Lemma 5 is proven. ■

These lemmas provide the proof of the general possibility theorem in the case of symmetric election procedures.

Theorem 3 If there are at least three candidates, then there is no REP which is symmetric with respect to both voters and candidates except complete indifference.

Proof: With Lemma 5, the preference schedules

$$A \succ B \succ C \qquad B \succ C \succ A \qquad C \succ A \succ B$$

provide APB, BPC, and CPA; thereby violating transitivity of the REP. Since there is no REP which works if individual preferences are restricted to strict preferences, there cannot be one for general individual preferences. This example is easily extended to show there cannot be a REP with any number of voters. ■

In the case of only two candidates, majority rule provides a REP for any number of voters. The proof is left as Exercises 5 and 6.

Election Procedures

Approval voting

The fact that there is no rational election procedure (which is non-dictatorial and non-imposed) if there are more than two candidates may have contributed to the pervasiveness of the two party system. But there is a rational election procedure which will produce a collective weak order of any number of candidates based on individual preferences consistent with PR and IA. However, although any number of candidates is allowed, all individual preference functions must be dichotomous: individuals either approve or disapprove of each candidate, but do not distinguish preferentially among the candidates in each category. The REP is simple — it is the rank order based on the number of approval votes each candidate receives.

Example 2 The operation of approval voting can be illustrated by a five candidate race (A, B, C, D, E). Each voter votes for whichever candidates he approves. For illustrative purposes assume that the ten ballots cast are

$$ABE,\ B,\ BCE,\ BD,\ AE,\ BCD,\ CE,\ BCDE,\ ABE,\ ABDE$$

(i.e., each voter lists the candidates of whom he approves; voter 1 approves Alfred, Barthowlamew, and Ethelred). Then A received 4 votes, B received 8 votes, C received 4 votes, D received 4 votes, and E received 7 votes. This provides the preferential order $B \succ E \succ A, C, D$. Ties (indifferences) are possible; it is not necessary that anyone receive approval from 50% of the voters, nor does receiving such approval assure election. □

The utility of approval voting is given in the following theorem.

Theorem 4 Approval voting provides a REP on the restricted domain where each voter has dichotomous preferences.

Proof: A dichotomous preference (i.e., a partition of the candidates into two sets: the "approved" set, whose members are related to every candidate, and the "nonapproved" set, whose members are only related to members of that set,) is a weak order. It must be shown that the resultant preference schedule is a weak order, and further that it satisfies properties PR and AI.

The vote tally identifies each candidate with an integer (the number of votes received). XRY if the number of votes X received is greater than or equal to the number of votes received by Y. The trichotomy for two integers a and b ($a > b$, $a = b$, or $b > a$) is equivalent to connectivity, and the transitive property of inequality is equivalent to transitivity; hence R is a weak order. Property PR is satisfied since giving an additional vote to a candidate will increase his vote count without affecting other vote counts, hence cannot result in his vote count becoming less than another candidate's. Property IA is satisfied since the relative ranking of two candidates depends solely on the number of votes they receive, i.e., is independent of the number of votes other candidates receive. Hence approval voting provides a REP if all individual preferences are dichotomous. ■

Example 3 Unfortunately, allowing individuals only the option of indicating approval or disapproval for each candidate significantly restricts their ability to express their preference. Consider the dilemma of an individual who, in a three candidate race, prefers A to B and prefers B to C (hence prefers A to C). It is clear that he should not vote for all three candidates or withhold his vote from all, because then his ballot would not affect the relative totals. He should vote for A because whatever the vote count is without his ballot, adding a vote to A can only serve his interest. Similarly he should not vote for C. But it is not clear whether he should vote for B. If without his ballot there is a tie between

B and C, then voting AB will serve his interest; but if there is a tie between A and B voting AB will retain the tie which he could have broken in his favor had he voted only A. If he does not know how others will vote, he does not know how to vote to favor his interests. □

Majority election procedures

Although approval voting produces a REP, it puts an unreasonable constraint on the individual preference functions. Since in the case of only two candidates majority rule provides a rational election procedure, it is natural to try to modify election procedures so that they entail only two candidates and majority rule will provide a rational election procedure. Are there satisfactory ways to dichotomize all elections, i.e., to make all elections a choice between two alternatives? The answer is no, but it is worth surveying some methods which have been employed in order to illustrate the problems which have manifested.

Sequential elections

Example 4 One possibility, perhaps more commonly used for passing legislation than for electing candidates, is to put two alternative choices to a vote at a time. For example, if there are three candidates A, B, and C; a vote could first be taken between A and B, and then between the winner of that contest and C. However, as the individual preferences of three voters ($A \succ B \succ C$), ($B \succ C \succ A$), and ($C \succ A \succ B$) illustrate, the ultimate winner could be different if the first vote were between B and C, followed by the winner of that contest versus A. □

Condorcet criterion

The paradox in Example 4 was noted by Marie-Jean-Antoine-Nicolas de Caritat, marquis de Condorcet*. Although election procedures are fraught with paradoxes, he concluded, based on two candidate results, that if any candidate would win all two way races, that person should be the winner. Such a person is called the **Condorcet** winner. If such a winner exists, the above paradox will not occur. It is easy to show that a Condorcet winner is unique, but such a winner need not exist. A. H. Copeland suggested generalizing the Condorcet

---

* Marie-Jean-Antoine-Nicolas de Caritat, marquis de Condorcet (1743–1794) was a protégé of Jean Le Rond d'Alembert. His *Essai sur l'application de l'analyse à la probabilité de décisions rendues à la pluralité des voix* (1785) assured him permanent place in the history of probability. He is also known as one of the first people to declare for a republic in the French revolution, but died an outlaw because he was too moderate when Robespierre came to power.

criterion by selecting as the winner the person who wins the most two way contests in the event that there is no Condorcet winner. Unfortunately, if there are fewer than five alternatives, there will never be a Copeland winner if there is not a Condorcet winner (see Exercise 22). Another method must be employed if one wishes to determine a winner.

Pluralities and runoffs

The essence of majority is maintained in plurality procedures, and runoff elections can be held in order to assure an actual majority. Runoffs may be held between the two recipients of the most votes (only first place votes count) or by sequentially eliminating the recipients of the fewest votes and transferring their votes to the next highest names on the respective ballots (these two methods are not equivalent).

Example 5 Although plurality decisions or runoff elections are widely used, they do not assure election of a Condorcet winner if one exists. In order to obtain a workable system, the most justifiable winner is often eliminated. The example with 9 voters, four with preference $(A \succ C \succ B)$, three with preference $(B \succ C \succ A)$, and two with preference $(C \succ B \succ A)$, illustrates that a Condorcet winner may lose either under plurality, a runoff between the top two vote-getters, or sequential runoffs eliminating the low vote-getter until somebody has a majority. (A voter whose top preference has been eliminated from a runoff votes for his most favored remaining candidate.) □

Applications to Sports

Most of the elections related to government are concerned only with determining a single winner rather than rank ordering the alternatives. But within the realm of our leisure activities there is a significant demand to for ranking alternatives. We conclude by considering some procedures employed to rank sports teams.

The Borda method

A vote count method to produce a collective ranking of alternative candidates based on individual rankings was proposed by Jean-Charles de Borda* in 1781. If there are n candidates, the Borda count assigns to the last name on each

* Jean-Charles de Borda (1733–1799) was a French mathematician best known for his work in fluid mechanics, navigation, and geodesy. While in the French navy he served on several scientific voyages, but also took part in the American Revolution and was captured by the British in 1782.

ballot 0 points, to the next-to-last name 1 point, . . . , and to the first name $n-1$ points. The winner is then determined by summing for each candidate the number of points from each ballot. This is the method used to rank the top twenty college football teams by polling sportswriters. It has the advantage of converting the weak orders of individuals into a collective weak order. It reflects the entire preference schedules, not just the top candidates on each ballot.

However, the use of numerical rankings quantifies the degree of preference, which Arrow felt could not be justified. A major problem is that it encourages insincere voting. If you know that there are two candidates for the best team, it will be in your interest to rank the one you favor first and the other one (although you sincerely believe it to be the second best team) last. This gives voters the ability to essentially blackball a candidate. A Condorcet winner will not necessarily win by this procedure; in fact, a team which receives a majority of the first place votes may not win.

Example 6 Suppose three sportswriters decide to rank the football teams of Dartmouth, Harvard, Brown, and Yale. They might cast the following ballots: $(B \succ H \succ D \succ Y)$, $(H \succ D \succ Y \succ B)$, and $(B \succ H \succ Y \succ D)$. Even though Brown received a majority of the first place votes, the point total for Harvard is seven, while the total for Brown is only six. □

Elimination tournaments

The national championship in collegiate basketball, unlike football, does not rely on a poll of sportswriters to determine the winner. Rather the "best" teams are paired off, with the winners subsequently paired against other winners. (These elimination trees can be found in newspapers during the NCAA playoffs.) Ultimately, one team wins the finals, and by transitivity is better than all the other teams in the tournament. But this method only picks a single winner; it is not clear whether the team defeated in the final game or the team defeated by the champion in the semi-final is the second best team; there are $n/2$ candidates remaining for the worst team. The amount of ambiguity can be diminished if there are consolation games, but too many games such as a round robin tournament could produce a violation of transitivity if teams did not perform consistently. (It is also possible that there is no transitive ranking; abilities to implement and defend against various styles of play may allow team A to consistently beat team B, team B to consistently beat team C, and team C to consistently beat team A.)

Example 7 It is desired to rank the teams RI, CT, NH, and VT. A tournament in which CT beats NH and RI beats VT in the first round followed by CT defeating RI in the final produces CT as the best team since it defeated NH and RI, and the RI victory over VT makes CT better than VT by transitivity.

Otherwise, all that has been demonstrated is that RI is better than VT; either VT or NH could be the worse team, either RI or or NH could be the second best team. A consolation game in which VT beat NH would establish the order $CT \succ RI \succ VT \succ NH$, but if NH beat VT it would not be determined which of RI and NH is better. □

Example 8 A round robin tournament of six games between four teams in which A beats B, B beats C, C beats D, D beats A, A ties C, and B ties D shows that transitivity need not hold. □

In summary, whether electing people who will determine the fate of the world or judging the entrants in a fiddle competition, there is no best procedure. All we can do is know the limitations of each of the various methods, and hope it does not matter.

Suggested Readings

1. K. Arrow, *Social Choice and Individual Values*, Second Edition, Yale University Press, New Haven, CT, 1970.

2. S. Brams, "Comparison voting" in S. Brams, W. Lucas, and P. Straffin, *Political and Related Models*, Springer-Verlag, New York, 1978, pp. 32–65.

3. S. Brams, and P. Fishburn, *Approval Voting*, Birkhäuser, Boston, 1983.

4. W. Lucas, "Social Choice: The Impossible Dream", *For All Practical Purposes*, ed. L. Steen, W. H. Freeman, New York, 1988, pp. 174–90.

5. R. Niemi and W. Riker, "The choice of voting system", *Scientific American*, Vol. 214, 1976, pp. 21–27.

6. P. Straffin, *Topics in the Theory of Voting*, Birkhäuser, Boston, 1980.

Exercises

1. Give an example, other than those discussed in the text, of a weak order which is not a total order.

2. Give an example, other than those discussed in the text, of a partial order which is not a weak order.

3. Show that connectivity implies reflexivity for all binary relations on a set A.

4. Show that indifference is an equivalence relation on a set A.

5. Show that the relation resulting from majority rule with just two candidates is connected and transitive, hence is a weak order.

6. Show that majority rule with just two candidates positively reflects individual preferences (PR) and is independent of irrelevant alternatives (IA), hence provides a REP.

7. Show that a total order cannot be an equivalence relation on a set with more than one element.

8. Prove Lemma 2: Given that properties PR and IA hold, if whenever XP_1Y and YP_2X, XPY; then whenever XP_1Y, XPY.

9. Prove Lemma 3 in the case that the REP is symmetric with respect to voters and candidates.

In Exercises 10–15, consider an election with three candidates in which the preferences of the nine voters are: $(A \succ B \succ C)$, $(A \succ C \succ B)$, $(A \succ C \succ B)$, $(A \succ C \succ B)$, $(B \succ C \succ A)$, $(B \succ C \succ A)$, $(B \succ C \succ A)$, $(C \succ B \succ A)$, and $(C \succ B \succ A)$.

10. Does any candidate have a majority of the first place votes? Which candidate wins by the plurality criterion?

11. Which candidate wins if there is a runoff between the top two candidates?

12. If every voter found only his favorite candidate acceptable, which candidate would win under approval voting? If every voter found his first two choices acceptable, which candidate would win under approval voting?

13. If there are sequential elections with the first election between A and B and then the winner of that contest versus C, which candidate will win?

14. Is there a Condorcet winner?

15. Which candidate wins by the Borda count method?

16. Sometimes the Borda Count method is modified to favor candidates who receive first place votes by awarding n points for a first place vote, $n-2$ points for a second place vote, and in general $n-m$ points for an mth place vote $(2 \leq m \leq n)$. What would be the ranking of the teams in Example 6 if this method were used?

17. In a round robin tournament (everyone plays every other team) with four teams A beats B and D, C beats A and B. What outcomes for B *vs.* D and C *vs.* D will provide a transitive ranking of the teams?

18. In a round robin tournament with n teams, what is the least number of games that can be played and violate transitivity?

19. Extend the example in the proof of Theorem 3 to show that there is no REP with any number of voters greater than two.

20. Continue the proof of Lemma 3 to show $BP_1A \Rightarrow BPA$ and $CP_1A \Rightarrow CPA$.

21. Show that $|\{i|XP_iY\}| > |\{i|YP_iX\}| \Rightarrow |XRY|$ using symmetry and PR.

22. Show that for three or four alternatives there cannot be a Copeland winner unless there is a Condorcet winner.

23. If there is a single elimination tournament with no consolation games for eight teams, how many total orderings of the teams will be consistent with the final outcome (i.e., how many total orderings are consistent with $A \succ B$, $C \succ D$, $E \succ F$, $G \succ H$, $A \succ C$, $E \succ G$, and $A \succ E$)?

24. If A can beat B and C, B can beat C and D, C can beat D, and D can beat A; how should a single elimination tournament be organized so that A will win? How should a single elimination tournament be organized so that A will not win?

Computer Projects

1. Write a computer program to determine the winner of an election if there is a runoff between the top two vote-getters (assuming no candidate had a majority of votes on the first ballot). The input will be the preference schedules of each voter (i.e., ordered n-tuples), with the first votes cast for the names on the top of the lists, and the second votes cast for the highest ranking of the remaining two candidates.

2. Write a computer program to determine the winner of an election if there are sequential runoffs with the lowest vote recipient eliminated each round until one candidate has a majority. The input will be the preference schedules of each voter (i.e., ordered n-tuples), with the first votes cast for the names on the top of the lists, and the subsequent votes cast for the highest ranking of the remaining candidates.

3. Write a computer program to implement approval voting (what form should the input be in?). (*Solution*: Input the names of the approved candidates, adding 1 to their vote count each time the name is entered, or enter Boolean vectors indicating approval or disapproval and sum them componentwise.)