Union Bounds on the Bit Error Probability of Coded MRC in Nakagami-m Fading

Salam A. Zummo, *Member, IEEE*

***Abstract*— In this letter new union bounds are derived for coded maximal ratio combining (MRC) over Nakagami-m fading channels. The union bounds are expressed in the product form, which makes them easily evaluated using the transfer function of the code. The bounds are general to any diversity order and coding scheme with a known transfer function. Results show that the new bounds are tight to simulation results for wide ranges of diversity orders and Nakagami parameters.**

***Index Terms*— Diversity, MRC, union bound, error probability, Rayleigh, Nakagami-m, fading, convolutional codes, TCM.**

I. Introduction

DIVERSITY is an effective technique to mitigate the effect of fading in wireless communication systems. The diversity gain is obtained by combining independently faded copies of the transmitted signal at the receiver. Among the diversity combining schemes are equal-gain combining (EGC), the generalized selection combining (GSC) and the maximal-ratio combining (MRC), in which the outputs of the matched filters of the diversity branches are summed after being weighted by the fading attenuation of each branch. The resultant signal-to-noise ratio (SNR) at the output of the combiner is the sum of the SNR's of the M branches.

The performance of coded MRC systems over Rayleigh fading channels was analyzed extensively in the literature [1]–[4]. In particular, the union bound of in [2] was represented in the product form which allows the use of the transfer function of the code. In [5], several bounds on the error probability of turbo codes over Rayleigh fading channels were presented. However, a more general statistical fading model is the Nakagami-m distribution [6]. Existing union bounds for coded MRC over Nakagami-m fading channels rely on the use of the integral representation of the $\text{erfc}(.)$ function, which results in bounds that are evaluated via numerical integration, see as an example [3], [7].

In this letter, we derive new union bounds on the bit error probability (BEP) of coded MRC systems over Nakagami-m fading channels. The bounds are presented in the product form allowing efficient computation of the bound using the transfer function of the code.

II. System Model

The transmitter in a coded system is generally composed of an encoder (e.g., convolutional, turbo, trellis-coded modulation (TCM), etc.), interleaver and a modulator. The encoder encodes a block of K information bits into a codeword of L symbols, $\mathbf{S} = \{s_l\}_{l=1}^{L}$. The code rate is $R_c = K/L$. Coherent reception is employed. Hence, the matched filter output of the i^{th} diversity branch for the l^{th} symbol in the codeword can be written as

$$y_{l,i} = \sqrt{E_s} a_{l,i} s_l + z_{l,i}, \quad (1)$$

where E_s is the average received signal energy per diversity branch, $\mathbf{a}_l = \{a_{l,i}\}_{i=1}^{M}$ are the fading amplitudes of the M diversity branches modeled as Nakagami random variables. Here, we assume ideal interleaving and independent and identically distributed (i.i.d.) diversity branches. The noise samples $\mathbf{z}_l = \{z_{l,i}\}_{i=1}^{M}$ are i.i.d. complex Gaussian random variables with zero-mean and noise variance of N_0.

III. Pairwise Error Probability

The pairwise error probability (PEP) is defined as the probability of decoding a codeword $\mathbf{S}$ as another codeword $\hat{\mathbf{S}}$. In the following the PEP is written in the product form as

$$P(\mathbf{S} \to \hat{\mathbf{S}}) = K_c \times \prod_{l=1}^{L} W(s_l, \hat{s}_l), \quad (2)$$

where $W(s_l, \hat{s}_l)$ is the error weight profile between $\hat{s}_l$ and s_l, and K_c is a tightening constant that does not depend on the error sequence. The case of $K_c = 1$ results in the Chernoff bound [8]. The form in (2) enables the use of the transfer function of the code to evaluate the union bound on the BEP.

The conditional PEP for MRC diversity can be written as

$$P(\mathbf{S} \to \hat{\mathbf{S}}|\mathbf{A}) = P\bigg(\sum_{l=1}^{L} \sum_{i=1}^{M} \Big(|y_{l,i} - \sqrt{E_s} a_{l,i} s_l|^2 - |y_{l,i} - \sqrt{E_s} a_{l,i} \hat{s}_l|^2 \Big) \geq 0 \Big| \mathbf{A} \bigg), \quad (3)$$

where $\mathbf{A}$ is a vector containing the fading gains of a codeword. In the following, we extend [2] to the Nakagami case. Defining $d_l = E_s|s_l - \hat{s}_l|^2/4N_0$ and $\gamma_l = \sum_{i=1}^{M} a_{l,i}^2$, the conditional PEP [2] becomes

$$P(\mathbf{S} \to \hat{\mathbf{S}}|\mathbf{A}) = \frac{1}{2} \text{erfc} \left(\sqrt{\sum_{l=1}^{L} d_l \gamma_l} \right). \quad (4)$$

Since the fading affecting different diversity branches are assumed to be i.i.d. and a_i's are Nakagami random variables, the probability density function (pdf) [6] of γ_l is given by

$$f_{\gamma_l}(\gamma) = \frac{m^{mM}}{\Gamma(mM)} \gamma^{mM-1} e^{-m\gamma}, \quad \gamma \geq 0, \ m \geq 0.5, \quad (5)$$

Manuscript received May 15, 2006. The associate editor coordinating the review of this letter and approving it for publication was Dr. Costas Georghiades. This work was supported by King Fahd University of Petroleum and Minerals (KFUPM), grant FT-2006-27.

S. Zummo is with the Electrical Engineering Dept., KFUPM, Dhahran 31261, Saudi Arabia (email: zummo@kfupm.edu.sa).

Digital Object Identifier 10.1109/LCOMM.2006.060735.

where $\Gamma(\cdot)$ is the Gamma function. The unconditional PEP is found by averaging (4) over the statistics of the γ_l's as

$$P(\mathbf{S} \rightarrow \hat{\mathbf{S}}) = \frac{1}{2}\int_0^\infty \cdots \int_0^\infty \operatorname{erfc}\left(\sqrt{\sum_{l=1}^{L} d_l\gamma_l}\right) \times f_\gamma(\gamma_1)\ldots f_\gamma(\gamma_L)d\gamma_1\ldots d\gamma_L. \quad (6)$$

Using the following change of variables

$$\delta_l = \frac{d_l}{1+d_l/m} \quad \text{and} \quad \omega_l = \gamma_l(1+d_l/m), \quad (7)$$

and re-arranging terms [2], the PEP becomes

$$P(\mathbf{S} \rightarrow \hat{\mathbf{S}}) = \frac{1}{2}\prod_{l\in\eta}\frac{1}{(1+d_l/m)^{mM}} \times \int_0^\infty \cdots \int_0^\infty \operatorname{erfc}\left(\sqrt{\sum_{l\in\eta}\delta_l\omega_l}\right)\exp\left[\sum_{l=1}^{L_\eta}\delta_l\omega_l\right] \times f_\omega(\omega_1)\ldots f_\omega(\omega_{L_\eta})d\omega_1\ldots d\omega_{L_\eta}, \quad (8)$$

where $\eta = \{l : s_l \neq \hat{s}_l\}$ and $L_\eta = |\eta|$ is the minimum time diversity of the code. In (8), the pdf's $f_\omega(\omega_l)$ follow the same form of (5) with ω_l replacing γ. Note that the variables $\{\omega_l\}$ that appear in (8) are different from those in (7). Define $\Omega = \sum_{l=1}^{L_\eta}\omega_l$, then the pdf of Ω is given by

$$f_\Omega(\Omega) = \frac{m^{mML_\eta}}{\Gamma(mML_\eta)}\Omega^{mML_\eta-1}e^{-m\Omega}, \quad \Omega \geq 0,\ m \geq 0.5. \quad (9)$$

Let $\delta_m = \min\{\delta_l, l \in \eta\}$, and note that $\sum_{l=1}^{L_\eta}\delta_l\omega_l \geq \delta_m\Omega$. Since $\operatorname{erfc}(\sqrt{x})e^x$ is a monotonically decreasing function for $x \geq 0$, then the PEP can be upper bounded as

$$P(\mathbf{S} \rightarrow \hat{\mathbf{S}}) \leq \frac{J}{2}\prod_{l=1}^{L_\eta}\left(\frac{1}{1+d_l/m}\right)^{mM}, \quad (10)$$

where

$$J = \frac{m^{mML_\eta}}{\Gamma(mML_\eta)}\int_0^\infty \operatorname{erfc}\left(\sqrt{\delta_m\Omega}\right)\Omega^{mML_\eta-1}e^{\Omega(\delta_m-m)}d\Omega. \quad (11)$$

In the following, the integral in (11) is simplified using two approaches resulting in two upper bounds on the PEP.

A. Bound 1

Using Eq. (6.286) of [9], the integral in (11) can be evaluated as

$$J = \frac{m^{mML_\eta}\Gamma(mML_\eta+0.5)}{\sqrt{\pi}mML_\eta\Gamma(mML_\eta)\delta_m^{mML_\eta}} \times {}_2F_1\Big(mML_\eta, mML_\eta+0.5; mML_\eta+1; 1-\frac{m}{\delta_m}\Big), \quad (12)$$

where ${}_2F_1(.,.;.;.)$ is the Gaussian confluent hypergeometric function [9]. Defining $x = 1 - \frac{m}{\delta_m}$ and using the relation ${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}\ {}_2F_1(\alpha,\gamma-\beta;\gamma;z/(z-1))$ results in

$$J = \frac{m^{mML_\eta}\Gamma(mML_\eta+0.5)}{\sqrt{\pi}\Gamma(mML_\eta+1)\delta_m^{mML_\eta}}(1-x)^{-mML_\eta} \times {}_2F_1\Big(mML_\eta, 0.5; mML_\eta+1; \frac{x}{x-1}\Big). \quad (13)$$

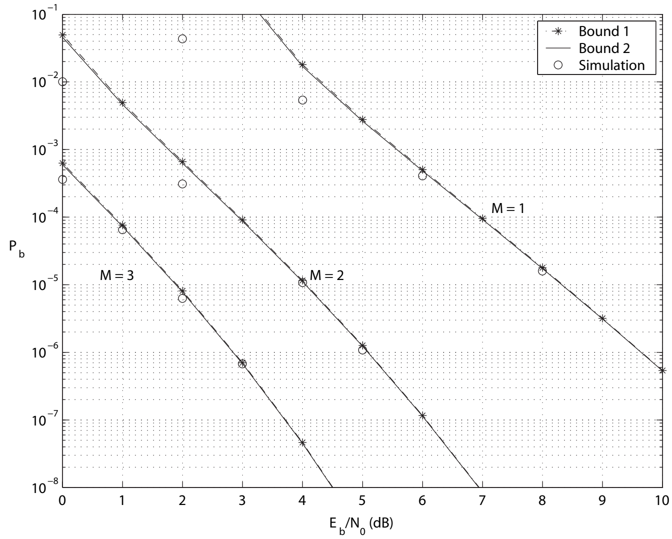

Fig. 1. BEP of a convolutionally coded MRC in Nakagami fading with $m = 2$ and different number of diversity branches.

Using the relation ${}_2F_1(\alpha,\gamma-\beta;\gamma;z) = \alpha z^{-\alpha}B_z(\alpha,\gamma-\beta)$ and substituting (13) in (10), the PEP can be simplified to

$$P(\mathbf{S} \rightarrow \hat{\mathbf{S}}) \leq \frac{\Gamma(mML_\eta+0.5)}{2\sqrt{\pi}\Gamma(mML_\eta)}x^{-mML_\eta}(x-1)^{mML_\eta} \times B_{x/(x-1)}(mML_\eta, 0.5)\prod_{l=1}^{L_\eta}\frac{1}{(1+d_l/m)^{mM}}, \quad (14)$$

where $B_x(.,.)$ is the incomplete Beta function [9]. Using the transfer function of the code, the BEP is upper bounded by

$$P_b \leq \frac{1}{k}\underbrace{\frac{\Gamma(mML_\eta+0.5)}{2\sqrt{\pi}\Gamma(mML_\eta)}x^{-mML_\eta}(x-1)^{mML_\eta}} \times \underbrace{B_{x/(x-1)}(mML_\eta, 0.5)}\frac{\partial T(D,I)}{\partial I}\Bigg|_{I=1, D=(1+\frac{d_l}{m})^{-mM}}, \quad (15)$$

where $T(D,I)$ is the transfer function of the code. Here, at each transition in the code trellis, the exponent of D represents the distance between the symbol label of the trellis transition and the symbol corresponding to the all-zero sequence, whereas the exponent of I represents the weight of the corresponding information sequence. Note that the underscored terms in (15) represent the tightening constant of Bound 1, i.e., the term K_c of (2).

B. Bound 2

Making the change of variable $\xi = \Omega(m-\delta_m)$ and using the integral form of the Q(.) function [10], the integral in (11) can be written for integer Nakagami parameter, m as

$$J = \frac{1}{(1-\delta_m/m)^{mML_\eta}}\int_0^\infty\left(\sqrt{\frac{2}{\pi}}\int_{\sqrt{2\nu\xi}}^\infty e^{-\tau^2/2}d\tau\right) \times \frac{\xi^{mML_\eta-1}e^{-\xi}}{(mML_\eta-1)!}d\xi, \quad (16)$$

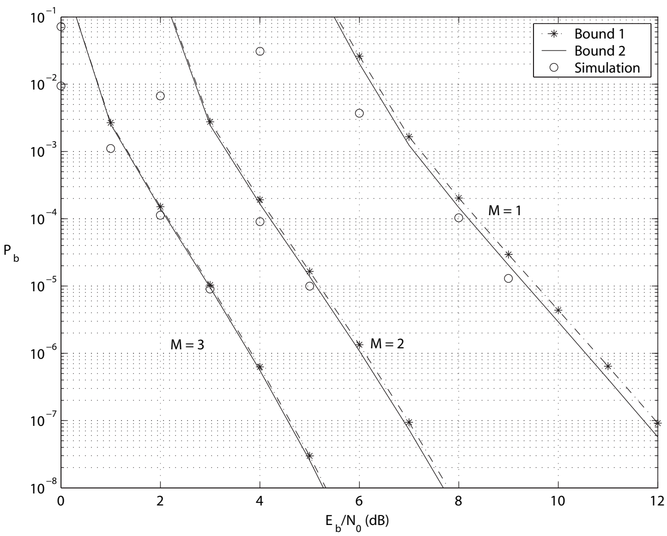

Fig. 2. BEP of 8PSK TCM coded MRC in Nakagami fading with $m = 4$ and different number of diversity branches.

where $\nu = \frac{\delta_m}{m-\delta_m}$. Changing the order of integration and using the properties of the number of arrivals in a Poisson random process as in [5], (16) simplifies to

$$J = \frac{\sqrt{2}}{\sqrt{\pi}(1-\delta_m/m)^{mML_\eta}} \int_0^\infty e^{-\tau^2/2} \times \left[\sum_{r=mML_\eta}^{\infty} \frac{1}{r!} e^{-\frac{\tau^2}{2\nu}} \left(\frac{\tau^2}{2\nu}\right)^r \right] d\tau, \tag{17}$$

which can be evaluated as

$$J = \frac{\sqrt{\delta_m/m}}{(1-\delta_m/m)^{mML_\eta}} \sum_{r=mML_\eta}^{\infty} \left(\frac{1-\delta_m/m}{4}\right)^r \binom{2r}{r}. \tag{18}$$

Following [5] and substituting (18) in (10), the PEP becomes

$$P(\mathbf{S} \to \hat{\mathbf{S}}) \le \frac{4^{-mML_\eta}}{2\sqrt{\delta_m/m}} \binom{2mML_\eta}{mML_\eta} \prod_{l=1}^{L_\eta} \frac{1}{(1+d_l/m)^{mM}}. \tag{19}$$

Using the transfer function of the code, the BEP is upper bounded by

$$P_b \le \frac{1}{k} \underbrace{\frac{4^{-mML_\eta}}{2\sqrt{\delta_m/m}} \binom{2mML_\eta}{mML_\eta}} \left.\frac{\partial T(D,I)}{\partial I}\right|_{I=1, D=(1+\frac{d_l}{m})^{-mM}}. \tag{20}$$

Note that the underscored term in (20) is the tightening constant of Bound 2, i.e., the term K_c of (2).

IV. Numerical Results

For illustration, the proposed union bounds were evaluated for a rate-1/2 (5,7) convolutionally coded BPSK and the 8-state 8PSK TCM designed in [8]. Figure 1 shows the performance of the convolutional code versus the SNR per information bit E_b/N_0 in dB, where $E_s = R_c E_b$. The performance of 8PSK TCM is shown in Figure 2. We observe that the new bounds are tight to simulation results for a wide range of SNR, diversity orders and Nakagami parameters.

V. Conclusions

Union bounds on the BEP of coherent coded MRC systems over Nakagami-m fading channel were derived. Results show that the bounds are tight to simulation results. Furthermore, proposed bounds are expressed in closed-forms that are simple to evaluate, unlike existing bounds which need numerical integration to be evaluated. Results show that the bounds are general to any coded system with a known transfer function over Nakagami-m fading with a general Nakagami parameter.

VI. Acknowledgements

The author acknowledges the support of KFUPM.

References

[1] J. Ventura-Traveset, G. Caire, E. Biglieri, and G. Taricco, "Impact of diversity reception on fading channels with coded modulation–part I: coherent reception," *IEEE Trans. Commun.*, vol. 45, pp. 563–572, May 1997.
[2] S. Al-Semari and T. Fuja, "Performance analysis of coherent TCM with diversity reception in slow Rayleigh fading," *IEEE Trans. Veh. Technol.*, vol. 48, pp. 198–212, Jan. 1999.
[3] A. Ramesh, A. Chockalingam, and L. Milstein, "Bounds on the performance of turbo codes on Nakagami fading channels with diversity combining," in *Proc. Global Telecommunications Conference (GLOBECOM) 2001*, vol. 2, pp. 1199–1204.
[4] I. Sason and S. S. (Shitz), "On improved bounds on the decoding error probability of block codes over interleaved fading channels, with applications to turbo-like codes," *IEEE Trans. Inf. Theory*, vol. 47, pp. 2275–2299, Sept. 2001.
[5] E. Ince, N. Kambo, and S. Ali, "Efficient expression and bound for pairwise error probability in Rayleigh fading channels, with application to union bounds for turbo codes," *IEEE Commun. Lett.*, vol. 19, pp. 25–27, Jan. 2005.
[6] M. Nakagami, "The *m*-distribution: a general formula of intensity distribution of fading," ser. in *Statistical Methods in Radio Wave Propagation*, W. C. Hoffman (editor). New York: Pergamon Press, 1960.
[7] M. Alouini and M. Simon, "A unified approach to the performance analysis of digital communication over generalized fading channels," in *Proc. of the IEEE*, vol. 86, pp. 1860–1877, Sept. 1998.
[8] S. H. Jamali and T. Le-Ngoc, *Coded-Modulation Techniques for Fading Channels*. Boston: Kluwer Academic Publisher, 1994.
[9] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products, Fourth Edition*. London: Academic Press Inc., 1965.
[10] M. Simon and D. Divsalar, "Some new twists to problems involving the Gaussian probability integral," *IEEE Trans. Commun.*, vol. 46, pp. 200–210, Feb. 1998.