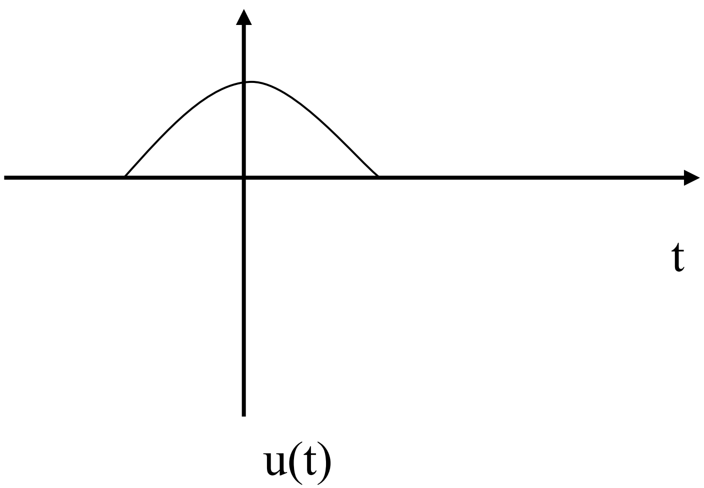
Time Invariance

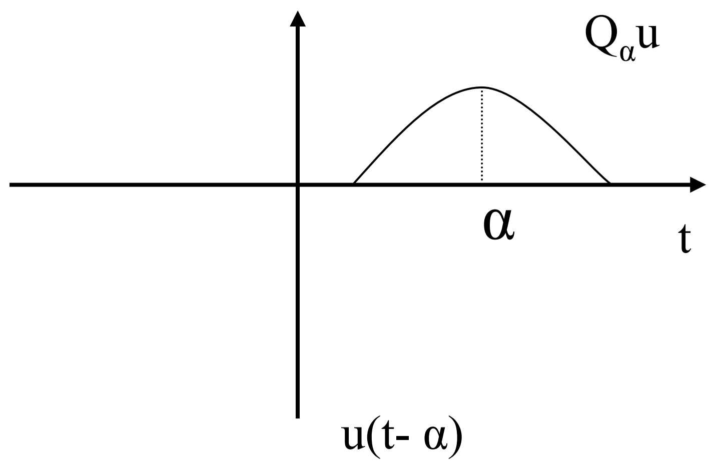
A system is said to be **time-invariant** if a time shift in the input signal causes a corresponding time shift in the output signal

To make the concept more precise, for fixed $\alpha \in R$, we introduced the shift operator

$$\mathbf{Q_\alpha u(t) = u(t - \alpha)}$$

Example

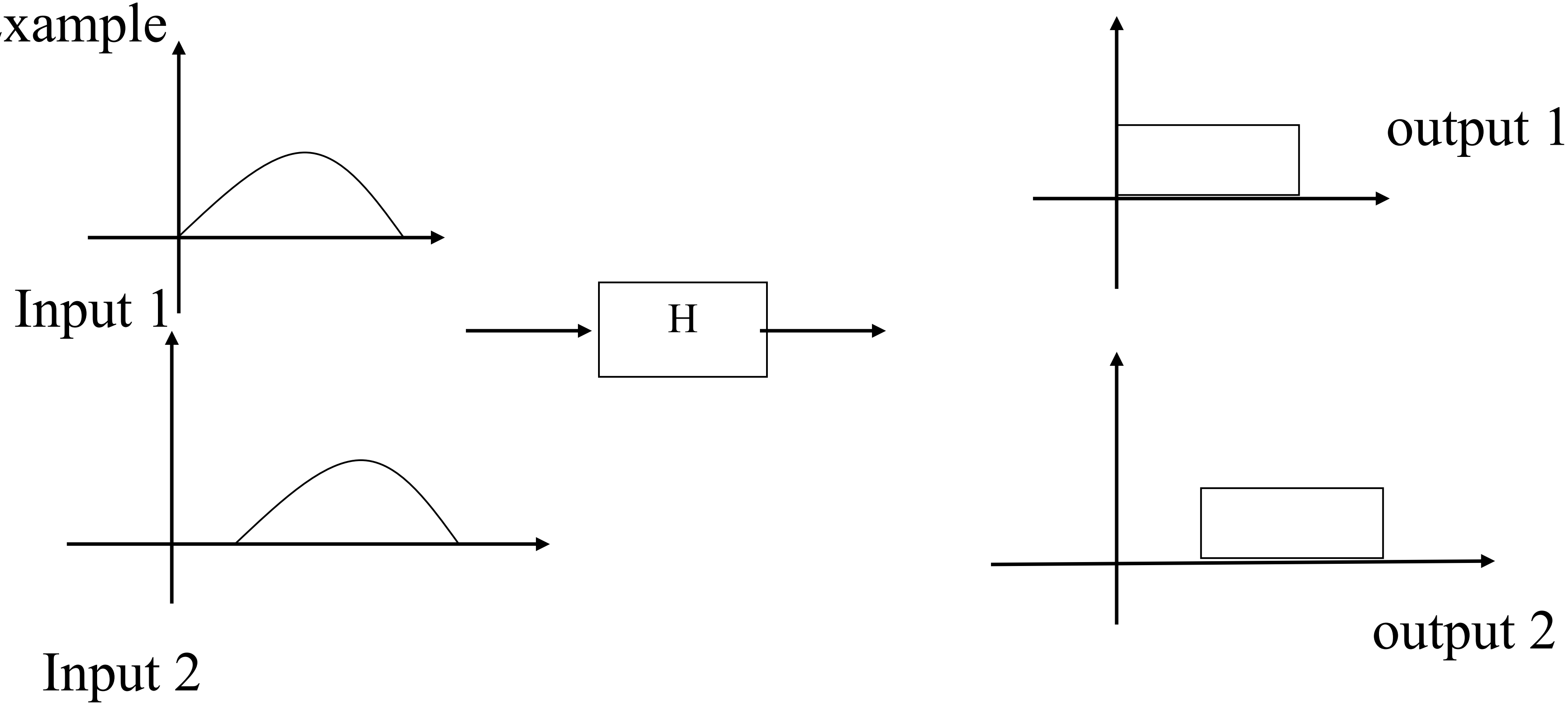




Definition: A system that is represented by the input output mapping y=H(u) is said to be time-invariant if and only if

$$H\,Q_\alpha(u) = Q_\alpha(H(u)) = Q_\alpha(y) \qquad \forall\ \alpha \in R \text{ and } \forall\ u$$

Example



Definition:

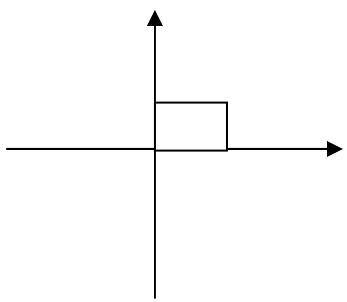
A system which is not time-invariant is said to be time-varying

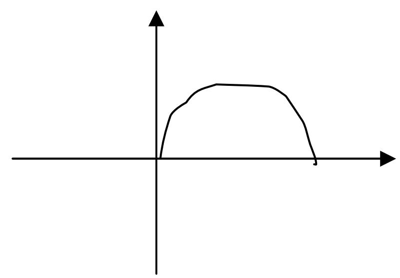
Example:

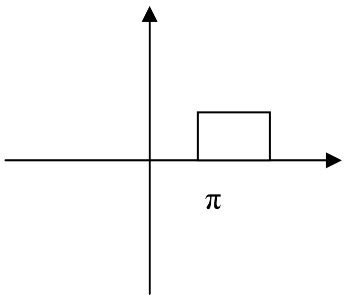
A system's Input/Output (I/O) mapping is given by

$$y(t) = \sin(t).u(t) \qquad \forall t$$

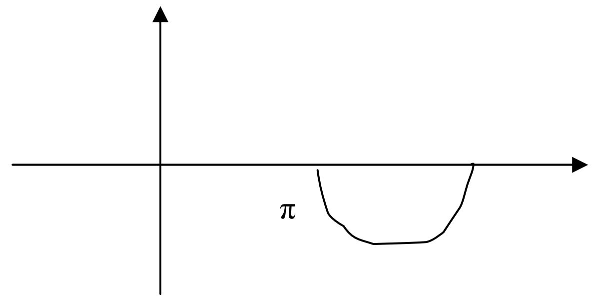
This system is time-varying. In fact







$Q_\pi u$



$Q_\pi y$

Given a linear system modeled by *H*. Then

$$(Hu)(t) = \int_{-\infty}^{\infty} g(t,\tau)u(\tau)d\tau$$

Question:

If *H* is also assumed to be time-invariant (in addition to linearity), what structure will time-invariance impose on $g(t,\tau)$?

Answer:

By time invariance,

$$g(t+\alpha,\tau+\alpha)=g(t,\tau) \quad \forall\alpha \ \forall t,\tau$$

Equivalently, $Q_{\alpha} g(.,\tau) = g(.,\tau+\alpha)$

As a result, for any given t,τ letting $\alpha=-\tau$

We have $g(t,\tau)=g(t+\alpha,\tau+\alpha)=g(t-\tau,0)$

Summarizing

For a linear time invariant system

$$(Hu)(t) = \int_{-\infty}^{\infty} \tilde{g}(t-\tau)u(\tau)d\tau$$

where $\tilde{g}(.) = g(.0) = \text{impulse respons}$

If the system is, in addition causal and is relaxed at $t = t_o$, then

$$(Hu)(t) = \int_{t_o}^{t} \tilde{g}(t-\tau)u(\tau)d\tau \quad t \geq t_o$$

Linear Time-Invariant Systems in the Frequency Domain

The output of a Linear Time-Invariant System (LTI) which is relaxed at $t=0$ is given by

$$y(t)=\int_0^\infty \tilde{g}(t-\tau)u(\tau)d\tau \quad t\geq 0$$

Taking the Laplace transform of both sides, we have:

$$\Im(y)=\hat{y}$$

$$\hat{y}(s)=\int_0^\infty y(t)e^{-st}dt$$

$$=\underbrace{\int_0^\infty \int_0^\infty \tilde{g}(t-\tau)u(\tau)d\tau}_{y(t)}\, e^{-st}dt$$

$$= \int_0^\infty \int_0^\infty \tilde{g}(t-\tau) e^{-st} u(t)\, d\tau dt$$

$$= \int_0^\infty \int_0^\infty \tilde{g}(t-\tau) e^{-st} dt u(\tau)\, d\tau$$

$$= \int_0^\infty \int_0^\infty \tilde{g}(t-\tau) e^{-s(t-\tau)} dt e^{-s\tau} u(\tau)\, d\tau$$

$$= \underbrace{\int_0^\infty \int_0^\infty \tilde{g}(v) e^{sv} dv}_{\hat{\tilde{g}}(s)} \; \mathrm{e}^{-st} u(\tau)\, d\tau$$

$$= \hat{\tilde{g}}(s) \int_0^\infty e^{-s\tau} u(\tau)\, d\tau$$

$$= \hat{\tilde{g}}(s) \hat{u}(s)$$

SISO Case:

$$\hat{y}(s)=\hat{\tilde{g}}(s)\hat{u}(s)$$

$\hat{\tilde{g}}(s)$ is the system transfer function. It has two interpretations:

1. $\hat{\tilde{g}}(s)$ is the Laplace transform of the system's impulse response.
2. $\hat{\tilde{g}}(s)=\hat{y}(s)/\hat{u}(s)$ where y is the output corresponding to the input u when the system is relaxed at $t=0$

MIMO Case:

$$\hat{y}(s)=G(s)\hat{u}(s)$$

$\hat{\tilde{G}}(s)$ is the transfer function matrix. It is the Laplace transform of the impulse response matrix.

Definition: A rational function g(s) is said to be proper if $g(\infty)$ is finite (zero or nonzero) constant.

It is said to be strictly proper if $g(\infty) = 0$.

Example:

$$g(s)=\frac{s^2}{s-1}$$ is not proper

$$g(s)=\frac{s^2}{s^2-s+2}$$ is proper

$$g(s)=\frac{s^2}{s^3-s}$$ is strictly proper

Remark

If $g(s) = N(s)/D(s)$

$g(s)$ is proper if and only if $\deg N(s) < \deg D(s)$