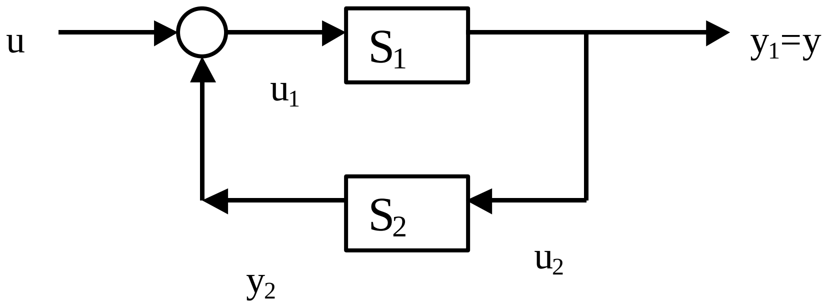
Feedback Connection
State Space Model



$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) \qquad \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t)$$
$$y_1(t) = C_1 x_1(t) + D_1 u_1(t) \qquad y_2(t) = C_2 x_2(t) + D_2 u_2(t)$$

but

$$u_1 = u \;\; y_2 = u \;\; C_2 x_2 - D_2 y_1$$
$$= u \;\; C_2 x_2 \;\; D_2(C_1 x_1 + D_1 u_1)$$

or

$$(I+D_2D_1)u_1 = u \;\; D_2C_1x_1 - C_2x_2$$

solving for u_1 gives

$$u_1 = (I+D_2D_1)^{-1}[u \;\; D_2C_1x_1 - C_2x_2]$$

$$\dot{x}_1 = [A_1 - B_1(I+D_2D_1)^{-1}D_2C_1]x_1 - B_1(I+D_2D_1)^{-1}C_2x_2 + B_1(I+D_2D_1)^{-1}u$$

$$y = y_1 = C_1x_1 + D_1u_1 = C_1x_1 + D_1(u - y_2)$$

$$= C_1x_1 + D_1u \;\; D_1(C_2x_2 + D_2y)$$

$$y = (I+D_1D_2)^{-1}[C_1x_1 + D_1u \;\; D_1C_2x_2]$$

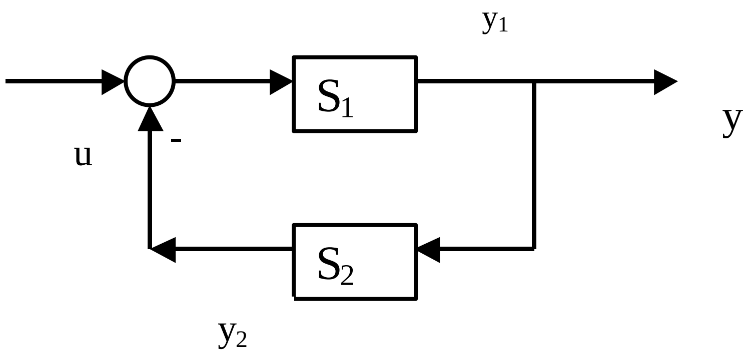
$$\dot{x}_2 = A_2 x_2 + B_2 y$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 - B_1 (I + D_2 D_1)^{-1} D_2 C_1 & -B_1 (I + D_2 D_1)^{-1} C_2 \\ B_2 (I + D_1 D_2)^{-1} C_1 & A_2 - B_2 (I + D_1 D_2)^{-1} D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \begin{bmatrix} B_1 (I + D_2 D_1)^{-1} \\ B_2 (I + D_1 D_2)^{-1} D_1 \end{bmatrix} u$$

$$y = \begin{bmatrix} (I + D_1 D_2)^{-1} C_1 & -(I + D_1 D_2)^{-1} D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (I + D_1 D_2)^{-1} D_1 u$$

Feedback Connection



$y(s) = G_1(s)u_1(s) = G_1(s)(u(s)- G_2(s)y(s))$

$(I+ G_1(s)G_2(s))y = G_1(s)u(s)$

We assume $(I+ G_1(s)G_2(s))^{-1}$ exist

(i.e. det $((I+ G_1(s)G_2(s))$

$y(s) = (I+ G_1(s)G_2(s))^{-1}G_1(s)u(s)$

or

$G(s) = (I+ G_1(s)G_2(s))^{-1}G_1(s) = G_1(I+ G_2(s)G_1(s))^{-1}$

Note that :

Making the assumption that det $((I+ G_1(s)G_2(s))$ essential for the closed loop mathematical formulation to make sense. To see this

Consider the example:

$$G_1(s) = \begin{bmatrix} \frac{1-s}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{-s}{s+1} \end{bmatrix} \qquad G_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I + G_1(s)G_2(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix}$$

And the det $(I + G_1(s)G_2(s)) = 0$

From the block diagram, $(I + G_1(s)G_2(s))y(s)=G_1(s)u(s)$

So

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1-1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{-s}{s+1} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$$

If

$$u(s) = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}, \text{ then we have } \begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1-s}{s^2} \\ \frac{1}{s^2} \end{bmatrix}$$

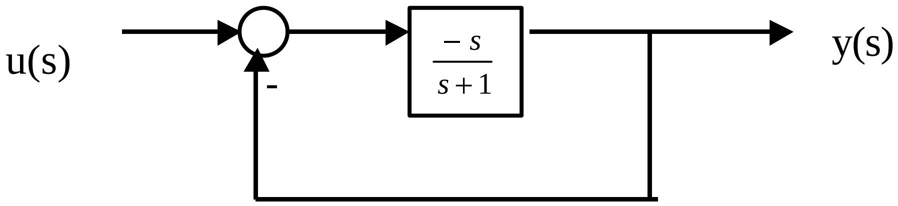
For which there is no solution

We have seen that det $((I+ G_1(s)G_2(s))$

essential.

Even when $((I+ G_1(s)G_2(s))^{-1}$ exists the transfer function from u(s) to another point in the loop may not be proper.

Example:



Here det $(I+ G_1(s)G_2(s))$ = 1+G(s) = 1+ -s/(s+1) = 1/(s+1)0

However

$$G(s) = \frac{\frac{-s}{s+1}}{1+\frac{-s}{s+1}} = \frac{-s}{s+1-s} = -s$$

Improper System

Improper transfer functions do not correspond to good systems.

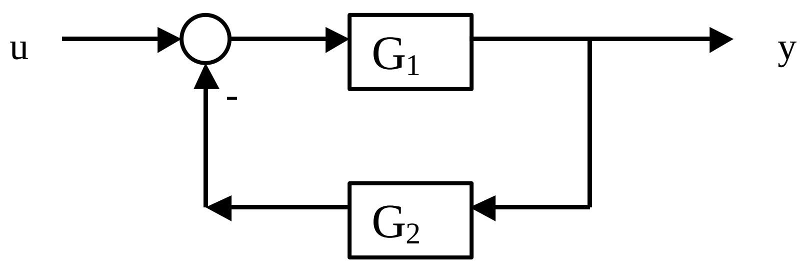
Problem??

NOISE

Well-posedness

Definition: Let every subsystem of a composite system be described by a rational transfer function. Than the composite system is said to be well posed if the transfer function of every subsystem is proper and the closed transfer function from any point chosen as an input terminal to every other point along the directed path is well defined and proper.

Theorem: Consider the feedback system



Let $G_1(s)$ and $G_2(s)$ be q p and p q proper rational transfer matrices. Than the overall transfer function

$$G(s) = G_1(s)(I+G_2(s)G_1(s))^{-1}$$

is proper if and only if $I+G_2($ $_1($

Discrete Time Systems

Inputs and outputs of discrete-time systems are defined only at discrete instants of time, t_o, t_1

instants of time are assumed to be an integral multiplies of some basic unit T, say

$t_o = 0$, $t_1 = T$, t_2

in which case T is often not explicitly shown and assumed that the time parameter, denoted by k, takes integral values,

so we define {y(k) = y(kT)} and {u(k) = u(kT)}

as the discrete output and input sequences.

For a linear relaxed discrete time system, we have

$$y(k) = \sum_{m=-\infty}^{\infty} g(k,m)u(m)$$

where *g(k, m)* is called the weighting sequence or the impulse response. It is the response to the input

$$d(n-m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

If the system is causal, and relaxed at k_o then we have

$$y(k) = \sum_{m=k_o}^{k} g(k,m)u(m)$$

If the system is time invariant and if we take $k_o = 0$, then

we have

$$y(k) = \sum_{m=0}^{k} g(k-m)u(m) \qquad *$$

Z Transform

The Z Transform of the sequence { }

is defined as

$$u(z) = \mathrm{Z}\{u(k)\} = \sum_{k=0}^{\infty} u(k) z^{-k}$$

If the Z transform is applied to * then

y(z) = G(z) u(z)

State Space Model

Time Varying

$x(k+1) = A(k)x(k) + B(k)u(k)$

$y(k) = C(k)x(k) + D(k)u(k)$

Time Invariant Systems

$x(k+1) = Ax(k) + Bu(k)$

$y(k) = Cx(k) + Du(k)$ **

Transfer Function from State Space

Let $X(z)$ be the Z-transform of $x(k)$

$$X(z) = Z\{x(k)\} = \sum_{k=0}^{\infty} x(k)z^{-k}$$

Let $x(0) = x_o$ then

$$Z\{x(k+1)\} = \sum_{k=0}^{\infty} x(k+1)z^{-k}$$

Let $m=k+1$

$$Z\{x(k+1)\} = \sum_{m=1}^{\infty} x(m) z^{-(m-1)} = z \sum_{m=1}^{\infty} x(m) z^{-m}$$

$$= z\{\sum_{m=0}^{\infty} x(m) z^{-m} - x(0)\}$$

$$= zX(z) \; - x_o$$

Applying z-transform to (**), gives

$zX(z) \; z \; x_o = Ax(z) + Bu(z)$

$y(z)=Cx(z)+Du(z)$

$x(z)= (zI - A)^{-1}x_o + (zI - A)^{-1}Bu(z)$

$y(z)= C[(zI - A)^{-1}x_o + (zI - A)^{-1}Bu(z)] + Du(z)$

If $x_o = 0$, then

$y(z) = (C(zI - A)^{-1}B + D)\, u(z)$

$\Longrightarrow \quad G(z) = C(zI - A)^{-1}B + D$