State-Variable Description

Motivation

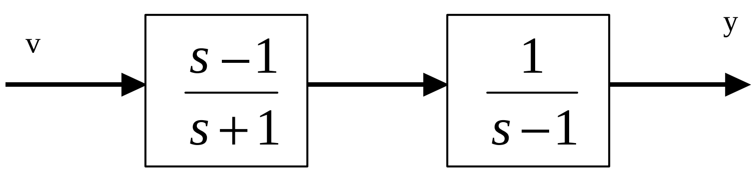
Consider a system with the transfer function

$$H_F(s) = \frac{1}{s-1}$$

Clearly the system is unstable

To stabilize it, we can precede $H_F(s)$ with a compensator

$$H_c(s) = \frac{s-1}{s+1}$$



The overall transfer function:

$$H_f(s)H_c = \frac{1}{s-1} \quad \frac{s-1}{s+1} = \frac{1}{s+1}$$

This is nice outcome, but unfortunately this technique will not work:

After a while the system will burn or saturate.

To see why, let us first set up an analog computer simulation of the cascade system

We can write the equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} v \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

There are general methods of solving such so-called state-space equation but it will suffice to proceed as follows:

The first equation is

$$\dot{x}_1 = -x_1 - 2v \quad , \quad x_1(0) = x_{10}$$

Which yields

$$x_1(t) = e^{-t} x_{10} \;\; 2e^{-t} * v$$

*denotes convolution

The second equation

$$\dot{x}_2 = x_1 + x_2 + v$$

has a solution

$$y(t) = x_2(t) = e^t x_{20} + \frac{1}{2}(e^t - e^{-t})\, x_{10} + e^{-t} * v$$

$$\Rightarrow y(s) = x_2(s) = \frac{x_{20}}{s-1} + \frac{x_{10}}{(s-1)(s+1)} + \frac{v(s)}{s+1}$$

Therefore the overall transfer function, which has to be calculated with zero initial condition is 1/(s+1) as expected.

Note: However, that unless the initial conditions can always be kept zero, y(.) will grow without bond.

So the input output description of a system is applicable only when the system is initially relaxed

State-Variable

Definition: The state of a system at time t_o is the amount of information at t_o that, together with $u_{[to,\infty)}$, determine uniquely the behavior of the system for all $t \geq t_o$

Usually x denotes state, u input, y output

Example

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} u(t)dt = \frac{1}{C}\int_{-\infty}^{to} u(t)dt + \frac{1}{C}\int_{to}^{t} u(t)dt$$

$$= y(t_o) + \frac{1}{C}\int_{to}^{t} u(t)dt$$

where

$$y(t_o) = \frac{1}{C}\int_{-\infty}^{to} u(t)dt$$

So if $y(t_o)$ is known, the output after $t \geq to$ can be uniquely determined. Hence, $y(t_o)$ regarded on the state at time t_o

Linearity

Definition: A system is said to be linear if for every t_o and any two state-input-output pairs

$$\left.\begin{array}{l} x_i(t_o) \\ u_i(t), \quad \mathrm{t} \geq \mathrm{t_o} \end{array}\right\} \rightarrow \mathrm{y_i}(t), \quad \mathrm{t} \geq \mathrm{t_o}$$

for i = 1, 2, we have

$$\left.\begin{array}{l} a_1 x_1(t_o) + a_2 x_2(t_o) \\ a_1 u_1(t) + a_2 u_2(t) \quad \mathrm{t} \geq \mathrm{t_o} \end{array}\right\} \rightarrow a_1 y_1(t) + a_2 y_2(t), \quad \mathrm{t} \geq \mathrm{t_o}$$

for any real constants ${}_1$ and ${}_2$. Otherwise the system is said to be nonlinear.

- Linearity must hold not only at the output but also at all state variables and must hold for zero initial state and nonzero initial state.
- This definition is different from

  $\mathbf{H({}_1u_1 + {}_2u_2) = {}_1\, H(u_1) + {}_2\, H(u_2)}$

Example

C and L are nonlinear

Because L C loop is in series connection with the current source, its behavior will not transmit to the output. Hence the above circuit is linear according the input-output definition while it is nonlinear according to the above definition of linearity.

- A very important property of any linear system is that the responses of the system can be decomposed into two parts

$$\text{Output due to } \begin{cases} x(t_o) \\ u(t), \qquad \mathrm{t} \geq \mathrm{t}_o \end{cases}$$

$$= \text{output due to } \begin{cases} x(t_o) \\ u(t) \equiv 0, \quad \mathrm{t} \geq \mathrm{t}_o \end{cases}$$

$$+ \text{ output due to } \begin{cases} x(t_o) = 0 \\ u(t), \qquad \mathrm{t} \geq \mathrm{t}_o \end{cases}$$

Or

Response = zero-input response + zero-state response

A very broad class of systems can be modeled by

$$\left.\begin{aligned}
\dot{x}_1 &= f_1(x_1,\ldots,x_n\ u_1,\ldots,u_p,t) \\
&\ \ \vdots \\
\dot{x}_n &= f_n(x_1,\ldots,x_n\ u_1,\ldots,u_p,t)
\end{aligned}\right] \Rightarrow \quad \dot{\underline{x}} = \underline{f}(\underline{x},\underline{u},t)$$

together with

$$\left.\begin{aligned}
y_1 &= g_1(x_1,\ldots,x_n\ u_1,\ldots,u_p,t) \\
&\ \ \vdots \\
y_q &= g_q(x_1,\ldots,x_n\ u_1,\ldots,u_p,t)
\end{aligned}\right] \Rightarrow \quad \underline{y} = \underline{g}(\underline{x},\underline{u},t)$$

where

$$\underline{x} : \begin{bmatrix} x_1 \\ \bullet \\ . \\ \bullet \\ x_n \end{bmatrix}, \underline{u} = \begin{bmatrix} u_1 \\ \bullet \\ . \\ \bullet \\ u_p \end{bmatrix}, \ and \ \underline{y} = \begin{bmatrix} y_1 \\ \bullet \\ . \\ \bullet \\ y_q \end{bmatrix}$$

We have seen an important special case, where

$$\underline{f}(\underline{x}, \underline{u}, t) \quad = \quad A(t)\,\underline{x} + B(t)\,\underline{u}$$

and

$$\underline{g}(\underline{x}, \underline{u}, t) \quad = \quad C(t)\,\underline{x} + D(t)\,\underline{u}$$

Fact (Existence & uniqueness)

Under some mild conditions on f(., ., .), the value of *x*(.) at t_o qualifies

e t_o, i.e. knowledge of $x(t_o)$. And *u*(t)

for $t \geq t_o$ gives a unique $\{y(t) : t\ ^3\ t_o\}$ & $\{x(t) : t\ ^3\ t_o\}$

Which solves the equations:

$$\dot{\underline{x}} = \underline{f}\ (\underline{x}, \underline{u}, t) \qquad\qquad \underline{y} = \underline{g}\ (\underline{x}, \underline{u}, t)$$

For the special cases:

$$\dot{x} = A(t)x + B(t)u$$
$$y = c(t)x + D(t)u$$

A sufficient condition for the existence of a unique solutions x(t), y(t) for $t \geq t_o$ given $x(t_o)$ and u(t), $t \geq t_o$ is that A(.) be a continues function.

We will make this assumption throughout the course.

Note: The above condition is always satisfied when A(.) is a constant matrix.