Power Spectrum of Digitally Modulated Signals

EE571
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Reading Material

- Proakis section 3.4

PSD of Digitally Modulated Signals with Memory

- Let *v(t)* be the bandpass modulated signal, its lowpass equivalent signal is :

$$v_l(t) = \sum_{n=-\infty}^{\infty} s_l(t - nT; I_n)$$

- Where $\mathbf{I_n}$ is the information sequence
- The autocorrelation of $v_l(t)$ is

$$R_{v_l}(t+\tau, t) = E\left[v_l(t+\tau)v_l^*(t)\right]$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[s_l(t+\tau-nT; I_n)s_l^*(t-mT; I_m)\right]$$

PSD of DM Signals with Memory

- $v_l(t)$ is cyclostationary since changing t to t+T does not change the mean and autocorrelation function.
- To determine its power spectral density, we need to average over one period T.
- Let k = n – m

$$\overline{R}_{v_l}(\tau) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_0^T E\left[s_l(t+\tau-mT-kT; I_{m+k})s_l^*(t-mT; I_m)\right]dt$$

Change *u=t-mT*

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-mT}^{-(m-1)T} E\left[s_l(u+\tau-kT; I_k)s_l^*(u; I_0)\right]du$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[s_l(u+\tau-kT; I_k)s_l^*(u; I_0)\right]du$$

PSD of DM Signals with Memory

Let $g_k(\tau) = \int_{-\infty}^{\infty} E\left[s_l(t+\tau; I_k) s_l^*(t; I_0) \right] dt$

Then

$$\bar{R}_{v_l}(\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} g_k(\tau - kT)$$

And the PSD of $v_l(t)$ is

$$S_{v_l}(f) = \frac{1}{T} F\left[\sum_k g_k(\tau - kT) \right]$$
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} G_k(f) e^{-j2\pi kfT}$$

PSD of DM Signals with Memory

Where $G_k(f)$ is the Fourier transform of $g_k(\tau)$

$$G_k(f) = F\left[\int_{-\infty}^{\infty} E\left[s_l(t+\tau; I_k) s_l^*(t; I_0) \right] dt \right]$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E\left[s_l(t+\tau; I_k) s_l^*(t; I_0) \right] e^{-j2\pi f\tau} \, dt \, d\tau$$
$$= E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_l(t+\tau; I_k) e^{-j2\pi f(t+\tau)} s_l^*(t; I_0) e^{j2\pi ft} \, dt \, d\tau \right]$$
$$= E\left[S_l(f; I_k) S_l^*(f; I_0) \right]$$

PSD of DM Signals with Memory

- Note that $G_0(f) = E\left[\left|S_l(f;I_0)\right|^2\right]$ is real and $G_{-k}(f) = G_k^*(f)$
- Define $G_k'(f) = G_k(f) - G_0(f)$ Thus $G_{-k}'(f) = G_k'^*(f)$, $G_0'(f) = 0$
- Then we can write

$$S_{v_l}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\left(G_k(f) - G_0(f)\right)e^{-j2\pi kfT} + \frac{1}{T}\sum_{k=-\infty}^{\infty}G_0(f)e^{-j2\pi kfT}$$

- Since $\sum_{k=-\infty}^{\infty} e^{-j2\pi kfT} = \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f - \frac{k}{T}\right)$
- We can write

$$S_{v_l}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\left(G_k(f) - G_0(f)\right)e^{-j2\pi kfT} + \frac{1}{T^2}\sum_{k=-\infty}^{\infty}G_0(f)\delta\left(f - \frac{k}{T}\right)$$

PSD of DM Signals with Memory

$$S_{v_l}(f) = \frac{2}{T}\mathrm{Re}\left[\sum_{k=-\infty}^{\infty}\left(G_k(f) - G_0(f)\right)e^{-j2\pi kfT}\right] + \frac{1}{T^2}\sum_{k=-\infty}^{\infty}G_0(f)\delta\left(f - \frac{k}{T}\right)$$

$$= S_{v_l}^c(f) + S_{v_l}^d(f)$$

Continuous   Discrete

PSD of Linearly Modulated Signals

- In linearly modulated signals, such as ASK, PSK and QAM, the lowpass equivalent of the modulated signal is

$$v_l(t) = \sum_{n=-\infty}^{\infty} I_n g(t-nT)$$

- Thus in this case, $s_l(t, I_n) = I_n g(t)$

- And

$$G_k(f) = E\left[S_l(f; I_k) S_l^*(f; I_0) \right]$$
$$= E\left[I_k I_0^* |G(f)|^2 \right]$$
$$= R_I(k) |G(f)|^2$$

Autocorrelation of the information sequence

FT of g(t)

PSD of Linearly Modulated Signals

- In this case, the PSD will be:

$$S_{v_l}(f) = \frac{1}{T} |G(f)|^2 \sum_{k=-\infty}^{\infty} R_I(k) e^{-j2\pi kfT}$$
$$= \frac{1}{T} |G(f)|^2 S_I(f)$$

FT of the modulation pulse

- Where

$$S_I(f) = \sum_{k=-\infty}^{\infty} R_I(k) e^{-j2\pi kfT}$$

Pulse Shaping

PSD of the discrete time random process $\{I_n\}$

Spectral Shaping

Precoding of information sequence

Precoding

- To control the PSD, we can employ a precoder of the form

$$J_n = I_n + \alpha I_{n-1}$$

- In General, we can introduce a memory of length L

$$J_n = \sum_{k=0}^{L} \alpha_k I_{n-k}$$

- The generated waveform will be:

$$v_l(t) = \sum_{k=-\infty}^{\infty} J_k g(t-kT)$$

- And the resulting PSD will be

$$S_{v_l}(f) = \frac{1}{T}|G(f)|^2 \left| \sum_{k=0}^{L} \alpha_k e^{-j2\pi kfT} \right|^2 S_I(f)$$

Example 3.4 – 1 (Proakis)

- For a binary communication system, $I_n=\pm 1$, with equal probability and I_n's are independent.
- Using a rectangular pulse $g(t) = \Pi\left(\frac{t}{T}\right)$ to generate the signal

$$v(t) = \sum_{k=-\infty}^{\infty} I_k g(t-kT)$$

- The PSD will be in this form

$$S_v(f) = \frac{1}{T}\left|T\mathrm{sinc}(Tf)\right|^2 S_I(f)$$

- To determine $S_I(f)$, we need to find $R_I(k)=E[I_{n+k}I_n^*]$, Since the sequence $\{I_n\}$ is independent:

$$R_I(k) = \begin{cases} E\left[|I|^2\right] = 1 & , k=0 \\ E\left[I_{n+k}\right]E\left[I_n^*\right] = 0 & , k \neq 0 \end{cases}$$

Example 3.4 – 1 (Proakis)

- Thus, $S_I(f) = \sum_{k=-\infty}^{\infty} R_I(k) e^{-j2\pi kfT} = 1$
- And $S_v(f) = T\text{sinc}^2(Tf)$
- A precoding of the form $J_n = I_n + \alpha I_{n-1}$ where α is real, would result in a PSD of the form

$$S_v(f) = T\text{sinc}^2(Tf)\left(1 + \alpha^2 + 2\alpha\cos(2\pi fT)\right)$$

Numerical Estimation of PSD

The Periodogram Method

Let $S_{xx}(f)$ be the PSD of a stochastic process X(t), the an estimate of the PSD could be calculated as:

$$\tilde{S}_{XX}(f) = \frac{1}{NT_s}\left|X_N(f)\right|^2$$

Where T_s is the sampling period and N is the number of observed samples.

Also $X_N(f)$ is the discrete Fourier transform of the observed data sequence

$$X_N(f) = T_s \sum_{n=0}^{N-1} X(n) e^{-j2\pi fnT_s}$$

The Periodogram Method

Matlab function periodogram

http://www.mathworks.com/help/signal/ref/periodogram.html#btt5c35-2

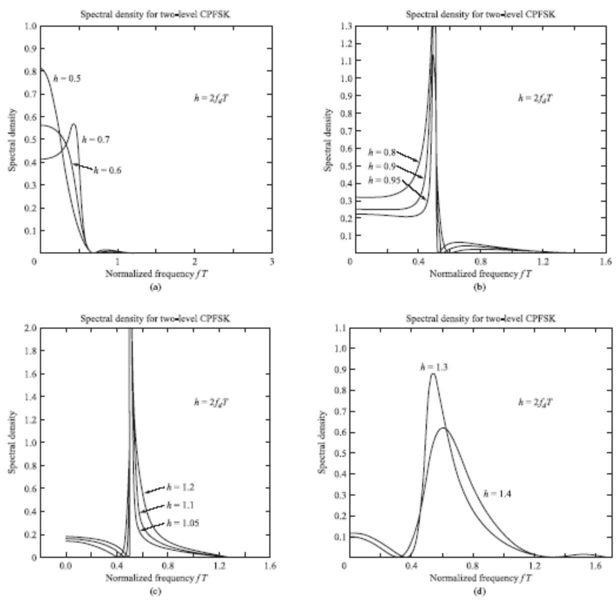

FIGURE 3.4–1
Power spectral density of binary CPFSK.

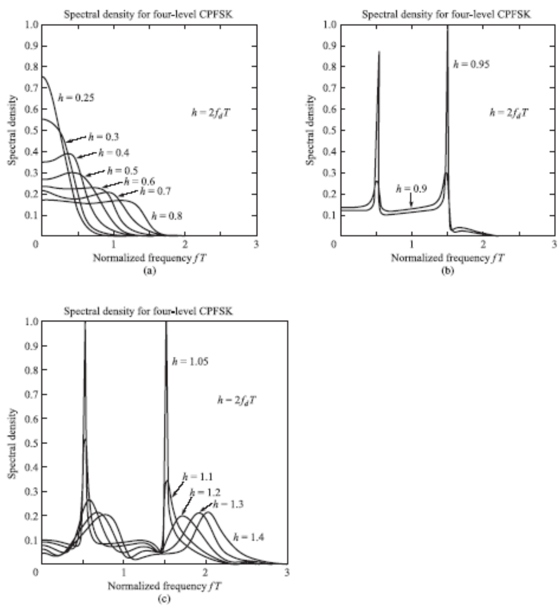

FIGURE 3.4–2
Power spectral density of quaternary CPFSK.

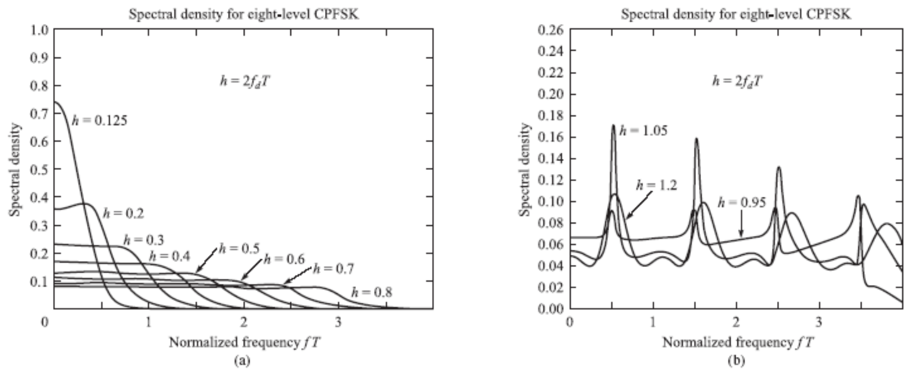

FIGURE 3.4–3
Power spectral density of octal CPFSK.

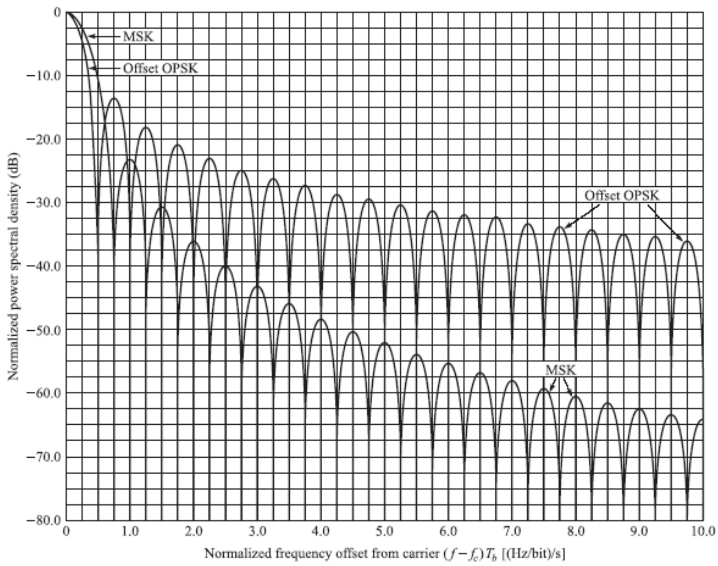

FIGURE 3.4–4
Power spectral density of MSK and OQPSK. [*Source:* Gronemeyer and McBride (1976);
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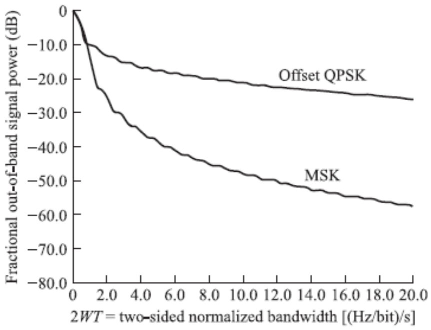

FIGURE 3.4–5
Fractional out-of-band power (normalized two-sided bandwidth = $2WT$). [*Source:* Gronemeyer and McBride (1976);
© IEEE.]

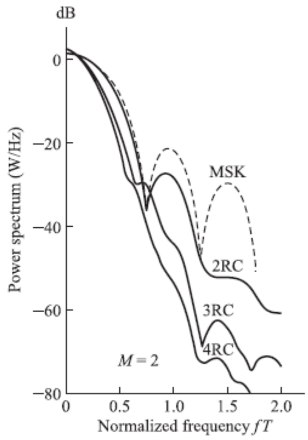

FIGURE 3.4–6
Power spectral density for binary CPM with $h = \frac{1}{2}$ and different pulse shapes. [*Source:* Aulin et al. (1981); © IEEE.]

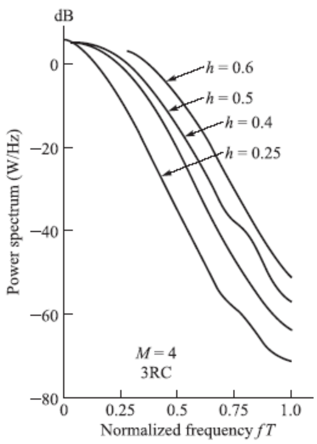

FIGURE 3.4–7
Power spectral density for $M = 4$ CPM with 3RC and different modulation indices. [*Source:* Aulin et al. (1981); © IEEE.]

Practice Problems

- Proakis 3.15, 3.16, 3.19, 3.24, 3.25, 3.26