EE 213 ELECTRIC CIRCUITS II

Resonance Circuits

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Recap: Sinusoidal Steady-State Analysis

- A sinusoidal voltage source is identified as:

$$v_s(t) = V_m \cos(\omega t + \phi)$$

where ω is the frequency, ϕ is the phase angle and V_m is the maximum amplitude.

- A sinusoidal current source is identified similarly as:

$$i_s(t) = I_m \cos(\omega t + \phi)$$

- The response of a **linear circuit** in steady-state in response to a sinusoidal source is also sinusoidal:

- ❑ with the **same frequency** and
- ❑ with possibly different amplitude and phase.

Recap: Phasor, Impedance & Admittance

- For linear circuits with sinusoidal sources, **phasor** transform leads to algebraic **frequency domain** analysis:

$$\mathcal{V} = \mathcal{V}_m e^{j\phi} = \mathcal{P}\{\mathcal{V}_m \cos(\omega t + \phi)\}$$

- The relation between voltage over an element and current through the element is expressed in the frequency domain as:

$$\mathcal{V} = \mathcal{Z}\mathcal{I}$$

- $\mathcal{Z}$ is referred to as the **impedance** and its reciprocal is called the **admittance**:

$$\mathcal{V} = \mathcal{I}/\mathcal{Y}$$

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Recap: Impedance and Related Values

Element	Impedance ($\mathcal{Z}$)	Reactance	Admittance ($\mathcal{Y}$)	Susceptance
Resistor	$\mathcal{R}$ (resistance)	–	$1/\mathcal{R}$ (conductance)	–
Inductor	$j\omega\mathcal{L}$	$\omega\mathcal{L}$	$1/(j\omega\mathcal{L})$	$-1/(\omega\mathcal{L})$
Capacitor	$1/(j\omega C)$	$-1/(\omega C)$	$j\omega C$	ωC

- All circuit analysis techniques for resistive circuits are extended to sinusoidal steady-state analysis by using the frequency-dependent impedance expressions.

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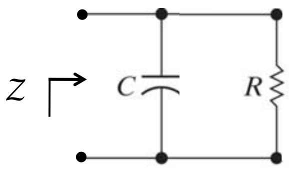
Frequency Response

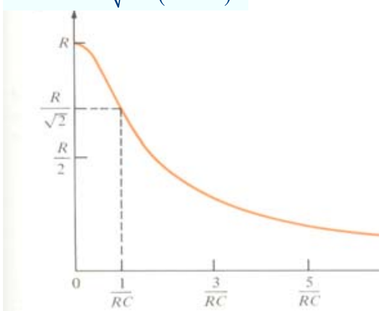
- The impedance of an arbitrary RLC circuit is thus **frequency-dependent**.
- This dependence can be depicted graphically by magnitude and phase angle versus frequency, which constitute the frequency response of the impedance.
- Parallel RC impedance:

$$Z = \frac{R\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{R}{1 + j\omega RC}$$



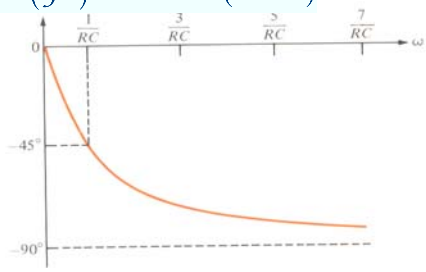
Frequency Response: Parallel RC

$$|Z(j\omega)| = \frac{R}{\sqrt{1 + (\omega RC)^2}}$$



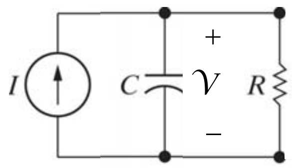
Amplitude response

$$\angle Z(j\omega) = -tan^{-1}(\omega RC)$$



Phase response

Half-Power or Cut-Off Frequency



$$|\mathcal{V}| = |\mathcal{Z}\mathcal{I}| = |\mathcal{Z}||\mathcal{I}|$$

- Cut-off frequencies are those at which the average power is half of its maximum value.
- Maximum average power for the above circuit:

$$|\mathcal{Z}| \text{ is max at } \omega = 0 \Rightarrow \mathcal{P}_{max} = \frac{1}{2}\frac{|\mathcal{V}_0|^2}{\mathcal{R}} = \frac{1}{2}\frac{|\mathcal{Z}(j0)|^2 |\mathcal{I}|^2}{\mathcal{R}} = \frac{1}{2}\mathcal{R}|\mathcal{I}|^2$$

- The power is halved at $\omega = \omega_c = 1/RC$:

$$\mathcal{P}_c = \frac{1}{2}\frac{|\mathcal{V}_c|^2}{\mathcal{R}} = \frac{1}{2}\frac{|\mathcal{Z}(j\omega_c)|^2 |\mathcal{I}|^2}{\mathcal{R}} = \frac{1}{4}\mathcal{R}|\mathcal{I}|^2 = \frac{1}{2}\mathcal{P}_0$$

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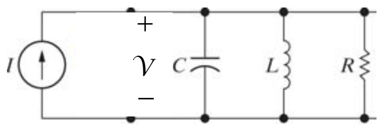
Resonance: Parallel RLC



- A circuit is said to be **at resonance** if the imaginary part of its impedance or admittance is zero.
- For the parallel RLC circuit:

$$\mathcal{Y} = \frac{\mathcal{I}}{\mathcal{V}} = \frac{1}{\mathcal{R}} + j\left(\omega C - \frac{1}{\omega \mathcal{L}}\right)$$

- The imaginary part (susceptance) is zero at:

$$\omega_r = \frac{1}{\sqrt{\mathcal{L}C}} \quad \text{: Resonance frequency}$$

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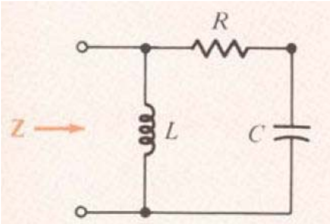
Example: Resonance Frequency



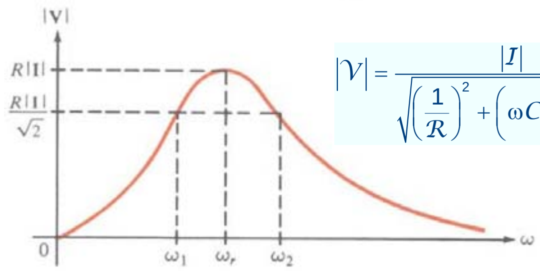
$$Z = \frac{\mathcal{V}}{\mathcal{I}} = \frac{j\omega\mathcal{L}\left(\mathcal{R} + \frac{1}{j\omega C}\right)}{j\omega\mathcal{L} + \mathcal{R} + \frac{1}{j\omega C}}$$

$$= \frac{j\omega\mathcal{L}(1 + j\omega\mathcal{R}C)}{1 - \omega^2\mathcal{L}C + j\omega\mathcal{R}C}$$

- The resonance frequency for this circuit is:

$$\omega_r = \frac{1}{\sqrt{\mathcal{L}C - \mathcal{R}^2C^2}}$$

Parallel RLC: Magnitude Response



$$|\mathcal{V}| = \frac{|\mathcal{I}|}{\sqrt{\left(\frac{1}{\mathcal{R}}\right)^2 + \left(\omega C - \frac{1}{\omega\mathcal{L}}\right)^2}}$$

- **Bandwidth** is defined in terms of the half-power frequencies as:

$$BW \equiv \omega_2 - \omega_1$$

- The response gets sharper as the bandwidth gets smaller.

Quality Factor

- The **quality factor** is defined at the resonance as:

$$\mathcal{Q} \triangleq 2\pi\left(\frac{\text{Maximum energy stored}}{\text{Total energy lost in a period}}\right)$$

- Since energy is stored by the capacitors/inductors and it is dissipated by the resistors, we have:

$$\mathcal{Q} = \frac{2\pi\, max_t\left[w_C(t)+w_L(t)\right]}{\mathcal{P}_R \mathrm{T}}$$

$$w_C(t) = \frac{1}{2}Cv_C^2(t)$$

$$w_L(t) = \frac{1}{2}\mathcal{L}i_L^2(t)$$

$\mathcal{P}_R$: Resistive power

Quality Factor: Parallel RLC

- Consider a source with current $i(t) = I\cos(\omega_r t)$
- At resonance, we have $\mathcal{Y} = 1/\mathcal{R}$. Hence:

$$v_C(t) = \mathcal{R}i(t) = \mathcal{R}I\cos(\omega_r t) \Rightarrow w_C(t) = \frac{1}{2}C\mathcal{R}^2I^2\cos^2(\omega_r t)$$

- The inductor current is obtained via phasor analysis:

$$I_L = \frac{\mathcal{R}I}{j\omega_r\mathcal{L}} = \frac{\mathcal{R}I}{\omega_r\mathcal{L}}e^{-j90^\circ} \Rightarrow i_L(t) = \frac{\mathcal{R}I}{\omega_r\mathcal{L}}\sin(\omega_r t)$$

- The energy stored in the inductor is then given by:

$$w_L(t) = \frac{1}{2}\frac{\mathcal{R}^2I^2}{\omega_r^2\mathcal{L}}\sin^2(\omega_r t) = \frac{1}{2}C\mathcal{R}^2I^2\sin^2(\omega_r t)$$

Quality Factor: Parallel RLC (ctd.)

- The total energy stored is thus a constant:

$$w_C(t) + w_L(t) = \frac{1}{2}\mathcal{C}\mathcal{R}^2\mathcal{I}^2$$

- The total energy dissipated by the resistor over a period is found as:

$$\mathcal{P}_{\mathcal{R}} = \frac{1}{2}\mathcal{I}^2\mathcal{R} \Rightarrow \mathcal{P}_{\mathcal{R}}\mathrm{T} = \mathcal{P}_{\mathcal{R}}\frac{2\pi}{\omega_r} = \frac{\pi\mathcal{I}^2\mathcal{R}}{\omega_r}$$

- Hence the quality factor of a parallel RLC circuit is:

$$\mathcal{Q} = \frac{\pi\mathcal{C}\mathcal{R}^2\mathcal{I}^2}{\frac{\pi\mathcal{I}^2\mathcal{R}}{\omega_r}} = \mathcal{R}\mathcal{C}\omega_r = \frac{\mathcal{R}}{\mathcal{L}\omega_r} = \mathcal{R}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}}$$

Bandwidth for Parallel RLC

- Since $\mathcal{Q}/\mathcal{R} = \omega_r\mathcal{C} = 1/\omega_r\mathcal{L}$, the admittance satisfies:

$$\mathcal{Y} = \frac{1}{\mathcal{R}} + j\left(\omega\mathcal{C} - \frac{1}{\omega\mathcal{L}}\right) = \frac{1}{\mathcal{R}}\left(1 + j\mathcal{Q}\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right)$$

- At the half-power frequencies, we have:

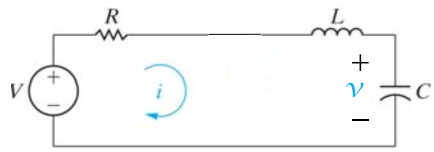
$$|\mathcal{V}| = \frac{\mathcal{R}|\mathcal{I}|}{\sqrt{2}} = \frac{|\mathcal{I}|}{|\mathcal{Y}|} \Rightarrow |\mathcal{Y}| = \frac{\sqrt{2}}{\mathcal{R}} \Rightarrow \mathcal{Q}\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right) = \mp 1$$

- We can obtain ω_1, ω_2 and the bandwidth as:

$$\omega_{1,2} = \omega_r\left(\sqrt{1 + \left(\frac{1}{2\mathcal{Q}}\right)^2} \pm \frac{1}{2\mathcal{Q}}\right) \Rightarrow \mathrm{BW} \triangleq \omega_2 - \omega_1 = \frac{\omega_r}{\mathcal{Q}} = \frac{1}{\mathcal{R}\mathcal{C}}$$

- The ratio $\mathcal{Q} = \frac{\omega_r}{\mathrm{BW}}$ is also called the **selectivity**.

Resonance: Series RLC

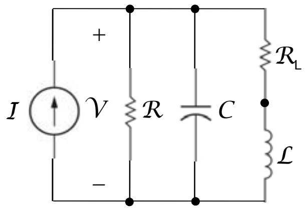


- Resonance frequency: $\omega_r = \dfrac{1}{\sqrt{LC}}$
- Quality factor: $Q = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$
- Half-power frequencies: $\omega_{1,2} = \omega_r\left(\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{1}{2Q}\right)$
- Bandwidth: $\mathrm{BW} = \dfrac{\omega_r}{Q} = \dfrac{R}{L}$

Resonance: Parallel and Series RLC

	Parallel RLC	Series RLC
Resonance frequency (ω_r)	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Quality factor (Q)	$= \omega_r RC = R\sqrt{\dfrac{C}{L}}$	$= \omega_r \dfrac{L}{R} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$
Half-power frequencies	$\omega_r\left(\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{1}{2Q}\right)$	$\omega_r\left(\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{1}{2Q}\right)$
Bandwidth (BW)	$= \dfrac{\omega_r}{Q} = \dfrac{1}{RC}$	$= \dfrac{\omega_r}{Q} = \dfrac{R}{L}$

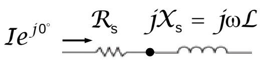
Series-Parallel RLC Circuit

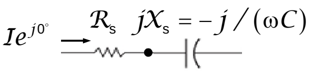


$$\mathcal{Y} = \frac{\mathcal{I}}{\mathcal{V}} = \frac{1}{\mathcal{R}} + j\omega C + \frac{1}{\mathcal{R}_L + j\omega\mathcal{L}} = \frac{1}{\mathcal{R}} + \frac{\mathcal{R}_L}{\mathcal{R}_L^2 + \omega^2\mathcal{L}^2} + j\left(\omega C - \frac{\omega\mathcal{L}}{\mathcal{R}_L^2 + \omega^2\mathcal{L}^2}\right)$$

- Resonance frequency: $\omega_r = \sqrt{\frac{1}{\mathcal{L}C} - \frac{\mathcal{R}_L^2}{\mathcal{L}^2}}$, for $\mathcal{R}_L \le \sqrt{\frac{\mathcal{L}}{C}}$
- No resonance for larger $\mathcal{R}_L$ values !
- For the parallel (series) RLC the magnitude of the admittance (impedance) is minimum. Neither is the case for the series-parallel RLC circuit.

Series Inductive or Capacitive Reactance





- The quality factor is defined in a similar way. Consider the case of inductive reactance.
- The maximum total energy stored:

$$w(t) = \frac{1}{2}\mathcal{L}i^2(t) = \frac{1}{2}\mathcal{L}\mathcal{I}^2\cos^2(\omega t) \Rightarrow w_{max} = \frac{1}{2}\mathcal{L}\mathcal{I}^2 = \frac{X_s}{2\omega}\mathcal{I}^2$$

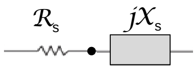
- The energy dissipated per period:

$$\mathcal{P}_s\mathrm{T} = \frac{1}{2}\mathcal{I}^2\mathcal{R}_s\left(\frac{2\pi}{\omega}\right) = \frac{\pi\mathcal{I}^2\mathcal{R}_s X_s}{\omega}$$

- Quality factor: $\mathcal{Q} = \frac{2\pi\mathcal{I}^2 X_s/(2\omega)}{\pi\mathcal{I}^2\mathcal{R}_s/\omega} = \frac{X_s}{\mathcal{R}_s}$

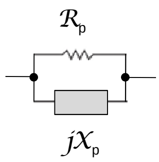
Series versus Parallel Reactance

- For a general **series** reactance:



$$Q = \frac{|X_s|}{R_s}$$

- For a general **parallel** reactance:

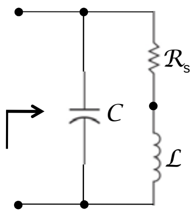


$$Q = \frac{R_p}{|X_p|}$$

- **Remark:** Note that the quality factor is frequency-dependent.

High-Q Resonant Circuit

$$Y = j\omega C + \frac{1}{R_s + j\omega L}$$



- Suppose that the series reactance has a large (≥ 20) Q-factor (called a high-Q coil).

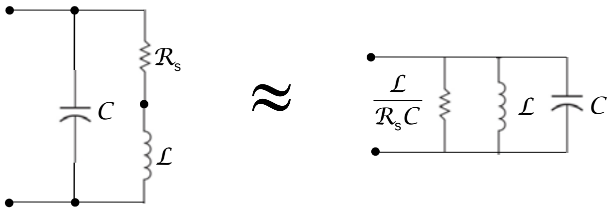
$$\frac{X_L}{R_s} \gg 1 \Rightarrow \frac{\omega L}{R_s} \gg 1 \Rightarrow \omega L \gg R_s$$

- The admittance can then be approximated by:

$$Y \approx j\omega C + \frac{1}{j\omega L}$$

- The resonance frequency is approximated as: $\omega_r \approx \frac{1}{\sqrt{LC}}$

Equivalent High-Q Resonant Circuit



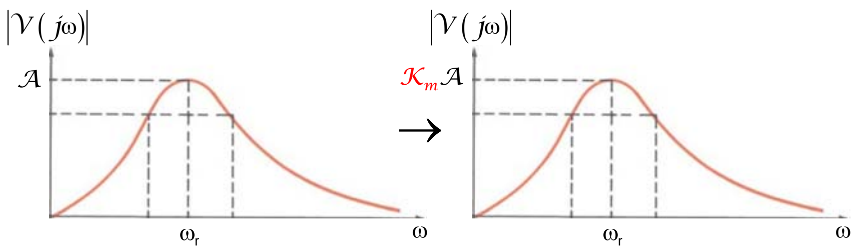
➢ Based on the impedance approximation

$$\mathcal{Z} = \frac{(\mathcal{R}_s + j\omega\mathcal{L})\dfrac{1}{j\omega C}}{\mathcal{R}_s + j\omega\mathcal{L} + \dfrac{1}{j\omega C}} \approx \frac{\mathcal{L}/C}{\mathcal{R}_s + j\omega\mathcal{L} + \dfrac{1}{j\omega C}}$$

the admittance can be approximated as

$$\mathcal{Y} \approx \frac{\mathcal{R}_s C}{\mathcal{L}} + j\omega C + \frac{1}{j\omega\mathcal{L}}$$

Magnitude Scaling of a Circuit

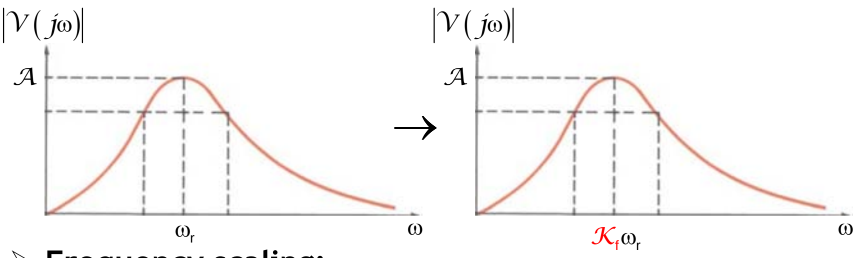


➢ **Magnitude (impedance) scaling:**

$$\mathcal{R} \rightarrow \mathcal{K}_m\mathcal{R} \qquad \mathcal{L} \rightarrow \mathcal{K}_m\mathcal{L} \qquad C \rightarrow \frac{C}{\mathcal{K}_m}$$

Dependent sources with a unit of Ω (Siemens) are multiplied (divided) by $\mathcal{K}_m$.

Frequency Scaling of a Circuit



- **Frequency scaling:**

$$\mathcal{R} \rightarrow \mathcal{R} \qquad \mathcal{L} \rightarrow \frac{\mathcal{L}}{\mathcal{K}_f} \qquad \mathcal{C} \rightarrow \frac{\mathcal{C}}{\mathcal{K}_f}$$

- Magnitude scaling does not change the bandwidth.
- Frequency scaling scales the bandwidth too:

$$\text{BW} \rightarrow \mathcal{K}_f \ \text{BW}$$