EE 213 ELECTRIC CIRCUITS II

CH 13
Complex Frequency and S-Domain

Dr. Hakan Köroğlu

King Fahd University of Petroleum & Minerals
ELECTRICAL ENGINEERING DEPARTMENT

Complex Frequency

- Phasor analysis can be generalized for linear circuits with damped-sinusoidal sources:

$$x(t) = X_m e^{\sigma t} \cos(\omega t + \phi)$$

$$\mathcal{P}\{X_m e^{\sigma t} \cos(\omega t + \phi)\} = X_m e^{j\phi};\ x(t) = \mathcal{Re}\{X_m e^{j\phi} \cdot e^{(\sigma + j\omega)t}\}$$

- The new ingredient σ (**nepers per sec.**) is called the **damping factor** and also as the **neper frequency**.
- The **complex frequency** is then defined as:

$$s = \sigma + j\omega$$

- The steady-state response of a linear circuit with damped-sinusoidal sources is also damped-sinusoidal with the **same** complex frequency.

Impedance and Admittances

- The relation between voltage and current is expressed in the complex frequency domain again as:

$$\mathcal{V} = \mathcal{Z}(s)\mathbf{I} = \mathbf{I} / \mathcal{Y}(s)$$

- Differentiation $\leftrightarrow$ multiplication with s in s-domain:

$$x(t) = \Re e\left\{X_m e^{j\phi} \cdot e^{st}\right\}; \ \dot{x}(t) = \Re e\left\{s X_m e^{j\phi} \cdot e^{st}\right\}$$

- The impedance and admittances of basic circuit elements are hence obtained in this case as:

Element	Impedance ($\mathcal{Z}$)	Admittance ($\mathcal{Y}$)
Resistor	$\mathcal{R}$	$1/\mathcal{R}$
Inductor	$s\mathcal{L}$	$1/(s\mathcal{L})$
Capacitor	$1/(s\mathcal{C})$	$s\mathcal{C}$

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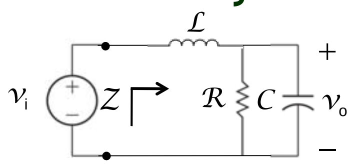
Example: Circuit Analysis in s Domain



- The impedance seen by the source is:

$$\mathcal{Z}(s) = s\mathcal{L} + \frac{\mathcal{R} \cdot 1/(s\mathcal{C})}{\mathcal{R} + 1/(s\mathcal{C})} = s\mathcal{L} + \frac{\mathcal{R}}{s\mathcal{R}\mathcal{C} + 1}$$

- The output voltage is related to the source voltage as:

$$\mathcal{H}(s) = \frac{V_o(s)}{V_i(s)} = \frac{\dfrac{\mathcal{R}}{s\mathcal{R}\mathcal{C}+1}}{\mathcal{Z}(s)} = \frac{\dfrac{1}{\mathcal{L}\mathcal{C}}}{s^2 + \dfrac{1}{\mathcal{R}\mathcal{C}} s + \dfrac{1}{\mathcal{L}\mathcal{C}}}$$

- Functions of this form are called **transfer functions**.

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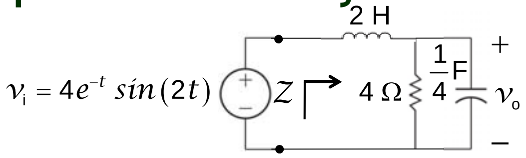
Example: Circuit Analysis in s Domain



- The output is obtained in the s-domain as:

$$\mathcal{V}_o = \underbrace{\left.\frac{2}{s^2 + s + 2}\right|_{s=-1+2j}}_{\mathcal{H}(-1+2j)} \cdot \underbrace{4e^{-j\frac{\pi}{2}}}_{\mathcal{V}_{si}} = \frac{1}{\sqrt{2}} e^{j\frac{3\pi}{4}} \cdot 4e^{-j\frac{\pi}{2}} = 2\sqrt{2}e^{j\frac{\pi}{4}}$$

- The forced response of the output is obtained as:

$$v_o(t) = \mathcal{Re}\left\{2\sqrt{2}e^{j45^\circ} \cdot e^{(-1+2j)t}\right\} = 2\sqrt{2}e^{-t} \cos\left(2t + \frac{\pi}{4}\right)$$

- **Remark**: The complete output response is obtained by adding the natural response.

Transfer Functions

- Linear time-invariant circuits are described by differential equations of the form:

$$a_n \frac{d^n v_o}{dt^n} + a_{n-1} \frac{d^{n-1} v_o}{dt^{n-1}} + \cdots + a_1 \frac{dv_o}{dt} + a_0 v_o = b_m \frac{d^m v_i}{dt^m} + \cdots + b_1 \frac{dv_i}{dt} + b_0 v_i$$

- Damped-sinusoidal inputs lead to damped-sinusoidal outputs of the same frequency in steady-state:

$$v_i(t) = \mathcal{V}_i e^{st} \Rightarrow v_o(t) = \mathcal{V}_o e^{st}$$

$$\left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right) \mathcal{V}_o e^{st} = \left(b_m s^m + b_{m-1} s^{n-1} + \cdots + b_1 s + b_0\right) \mathcal{V}_i e^{st}$$

- The transfer function from v_i to v_o is defined as:

$$\mathcal{H}(s) = \frac{\mathcal{V}_o e^{st}}{\mathcal{V}_i e^{st}} = \frac{\mathcal{N}(s)}{\mathcal{D}(s)} = \frac{b_m s^m + b_{m-1} s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Poles and Zeros

- The **zeros** of a transfer function $\mathcal{H}(s)$ are the values z_i for which $\mathcal{H}(z_i) = 0$.
- The **poles** of a transfer function $\mathcal{H}(s)$ are the values p_i for which $\mathcal{H}(p_i) = \infty$.
- Rigorous definitions are in terms of limits !
- A transfer function can then be expressed as:

$$\mathcal{H}(s) = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

- For real-valued coefficients, the complex-valued poles/zeros appear as conjugate pairs.
- Poles that are not **repeated** are called **simple**.

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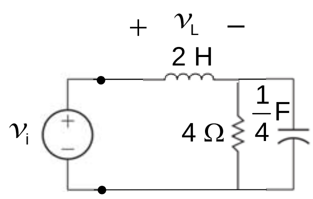
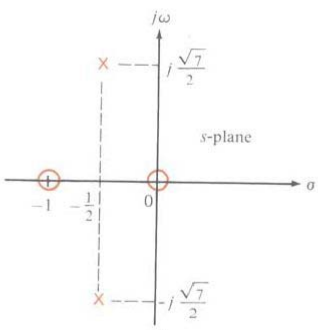
Pole-Zero Plot

- **Pole-Zero plot** is the depiction of poles as X and zeros as O in the complex number plane (or s-plane).
- **Example:** PZ plot of the transfer function from v_i to v_L





$$\mathcal{H}(s) = \frac{V_L}{V_i} = \frac{s(s+1)}{s^2+s+2}$$

$$= \frac{s(s+1)}{\left(s+0.5+j0.5\sqrt{7}\right)\left(s+0.5-j0.5\sqrt{7}\right)}$$

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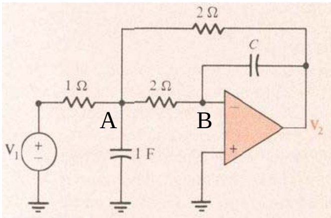
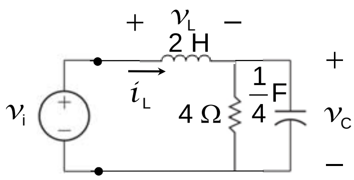
Example: An Op-Amp Circuit



- KCL at A: $\mathcal{V}_A - \mathcal{V}_1 + \frac{\mathcal{V}_A}{2} + \frac{\mathcal{V}_A - \mathcal{V}_2}{2} + s\mathcal{V}_A = 0 \Rightarrow (s+2)\mathcal{V}_A = \mathcal{V}_1 + \frac{\mathcal{V}_2}{2}$
- KCL at B: $\mathcal{V}_A = -\frac{s}{8}\mathcal{V}_2$
- Transfer function from v_1 to v_2:

$$\frac{\mathcal{V}_2}{\mathcal{V}_1} = -\frac{8}{s^2 + 2s + 4} = -\frac{8}{\left(s+1+j\sqrt{3}\right)\left(s+1-j\sqrt{3}\right)}$$

Poles and The Natural Response



- Consider the example circuit we have seen before:

$$\frac{\mathcal{V}_L(s)}{\mathcal{V}_i(s)} = \frac{s(s+1)}{s^2+s+2}, \quad \frac{\mathcal{V}_C(s)}{\mathcal{V}_i(s)} = \frac{2}{s^2+s+2}, \quad \frac{\mathcal{I}_L(s)}{\mathcal{V}_i(s)} = \frac{0.5(s+1)}{s^2+s+2}$$

- Observe the same poles in all the transfer functions!
- Provided that one portion of the circuit is not separated physically from the other, all transfer functions defined for it will have common poles.
- The poles determine natural response of a circuit and are hence also called **natural frequencies**.

Complete Response of a Linear Circuit

- Consider a damped sinusoidal (voltage or current) input to a linear time-invariant circuit:

$$u(t) = \mathcal{U} \cdot e^{st}, \quad \mathcal{U} = |\mathcal{U}| e^{j\angle \mathcal{U}}$$

- Any resulting voltage/current can be expressed as:

$$y(t) = \underbrace{y_{\text{na}}(t)}_{\text{Natural Response}} + \underbrace{y_{\text{fo}}(t)}_{\text{Forced Response}}$$

- The forced response is also damped-sinusoidal of the same (complex) frequency and is given by:

$$y_{\text{fo}}(t) = \mathcal{H}(s)\mathcal{U} \cdot e^{st}, \text{ if } s \text{ is not equal to a pole of } \mathcal{H}$$

where $\mathcal{H}(s)$ is the transfer function from u to y.

- The natural response is determined by the poles of the transfer function $\mathcal{H}(s)$ and the initial conditions.

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Natural Response of a Linear Circuit

- Consider a transfer function with simple poles:

$$\mathcal{H}(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)} = \frac{k(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}, \text{ m} < \text{n}; p_i \neq z_j, \forall \text{i,j}; \text{ i} \neq \text{j} \Rightarrow p_i \neq p_j$$

- The natural response of the output is of the form:

$$y_{\text{na}}(t) = \mathcal{A}_1 e^{p_1 t} + \cdots + \mathcal{A}_n e^{p_n t}$$

where $\mathcal{A}_i$'s are constants that are to be determined based on the initial conditions.

- When all poles are distinct except $p_i = p_{i+1} = \ldots = p_{i+k-1}$:

$$y_n(t) = \mathcal{A}_1 e^{p_1 t} + \cdots + \mathcal{A}_{i-1} e^{p_{i-1} t} + \left(\mathcal{A}_i + \mathcal{A}_{i+1} t + \cdots + \mathcal{A}_{i+k-1} t^{k-1}\right) e^{p_i t} + \mathcal{A}_{i+k} e^{p_{i+k} t} + \cdots + \mathcal{A}_n e^{p_n t}$$

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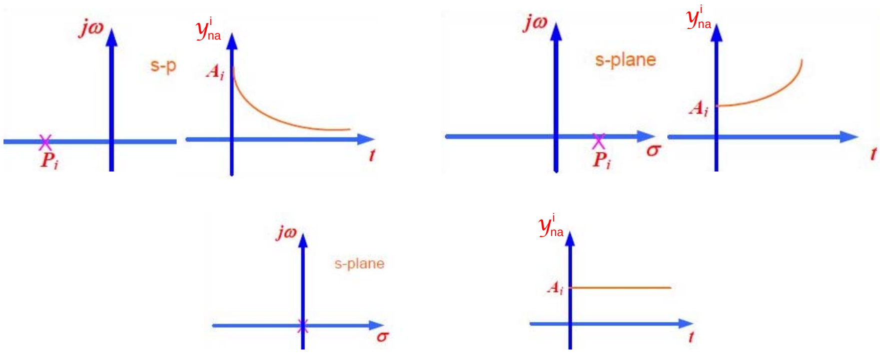
Natural Response: Distinct Real Poles



- Distinct real poles lead to responses in the form of exponential decay, exponential increase or constant response in time.

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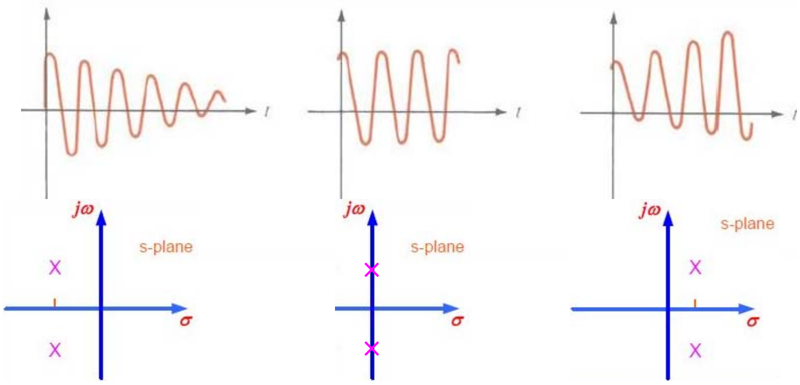
Natural Response:Distinct Complex Poles



- Distinct complex conjugate poles lead to oscillatory responses with exponentially decaying, exponentially increasing or constant amplitudes.

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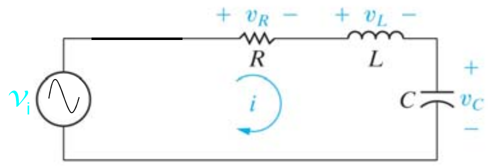
Recap: Series RLC Circuit



- The transfer function from ν_i to ν_C is given by:

$$\mathcal{H}(s) = \frac{\mathcal{V}_C}{\mathcal{V}_i} = \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2}, \quad \omega_n = \frac{1}{\sqrt{\mathcal{LC}}}, \quad \alpha = \frac{\mathcal{R}}{2\mathcal{L}}$$

- The poles of this transfer function are given by:

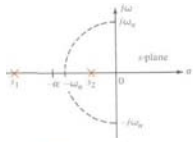
$$\begin{aligned} p_1 &= -\alpha + \sqrt{\alpha^2 - \omega_n^2} \\ p_2 &= -\alpha - \sqrt{\alpha^2 - \omega_n^2} \end{aligned}$$

Response of a Series RLC Circuit

$$\mathcal{H}(s) = \frac{\mathcal{V}_C}{\mathcal{V}_i} = \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2}, \quad \omega_n = \frac{1}{\sqrt{\mathcal{LC}}}, \quad \alpha = \frac{\mathcal{R}}{2\mathcal{L}}$$

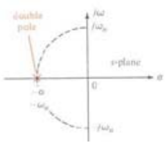
- Over-damped: $\alpha > \omega_n$

$$\nu_C(t) = \nu_C^{fo}(t) + \mathcal{A}_1 e^{p_1 t} + \mathcal{A}_2 e^{p_2 t}$$



- Critically-damped: $\alpha = \omega_n$

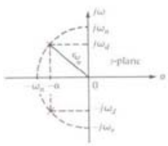
$$\nu_C(t) = \nu_C^{fo}(t) + (\mathcal{A}_1 + \mathcal{A}_2 t) e^{p_1 t}$$



- Under-damped: $\alpha < \omega_n$

$$\nu_C(t) = \nu_C^{fo}(t) + \mathcal{B} e^{\alpha t} \cos(\omega_d t + \phi),$$

where $\omega_d = \sqrt{\omega_n^2 - \alpha^2}$, $\mathcal{B} = 2|\mathcal{A}_1|$, $\phi = \angle \mathcal{A}_1$



Graphical Frequency Response Analysis

- Frequency response is obtained by replacing s with jω:

$$\mathcal{H}(s) = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \Rightarrow \mathcal{H}(j\omega) = k\frac{(j\omega-z_1)\cdots(j\omega-z_m)}{(j\omega-p_1)\cdots(j\omega-p_n)}$$

- Magnitude response:

$$|\mathcal{H}(j\omega)| = \frac{|k|\mathcal{N}_1\cdots\mathcal{N}_m}{\mathcal{D}_1\cdots\mathcal{D}_n}, \text{ where } \mathcal{N}_i = |j\omega - z_i|,\ \mathcal{D}_j = |j\omega - p_j|$$

- Phase response:

$$\angle\mathcal{H}(j\omega) = \phi_0 + \phi_1 + \cdots + \phi_m - \theta_1 - \cdots - \theta_n,$$

$$\text{where } \phi_0 = \angle k,\ \phi_i = \angle(j\omega - z_i),\ \theta_j = \angle(j\omega - p_j)$$

- Can be determined graphically by drawing the vectors jω – z_i and jω – p_j in the s-plane.

Example: Graphical Frequency Response

- Find $\mathcal{H}$(jω) for ω = 2:

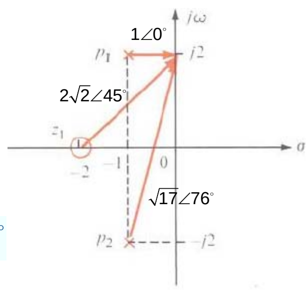
$$\mathcal{H}(s) = \frac{4(s+2)}{s^2+2s+5} = \frac{4(s+2)}{(s+1-2j)(s+1+2j)}$$

- Magnitude response:

$$|\mathcal{H}(2j)| = \frac{4\cdot 2\sqrt{2}}{1\cdot\sqrt{17}} = \frac{8}{17}\sqrt{34} \approx 2.7$$

- Phase response:

$$\angle\mathcal{H}(2j) = 0^\circ + 45^\circ - 0^\circ - 76^\circ = -31^\circ$$



- Bode plots display the responses in logarithmic scales and are rather easy to sketch.