Fading and MIMO Capacity

AWGN Real

$$Y = X + \eta \quad ; \text{where } \eta \sim (0, \sigma^2)$$

For Real signals and noise [One Dimensional]

$$C = \frac{1}{2}\log_2(1+SNR) = \frac{1}{2}\log(1+\rho)$$

where $SNR = \frac{E[X^2]}{E[\eta^2]} = \frac{P}{\sigma^2} = \rho$

AWGN Complex

$Y = X + \eta$; where η is a complex Gaussian noise with mean zero and variance $\frac{\sigma^2}{2}$ per dimension.

$$C = \log_2(1+SNR) = \log_2(1+\rho)$$

Flat Fading Rayleigh

$$Y = hX + \eta$$

where, η is a complex Gaussian noise with mean zero and variance $\frac{\sigma^2}{2}$ per dimension.

h is a complex Gaussian noise with mean zero and variance 0.5 per dimension.

- Let $h = \alpha + j\beta$

The envelope $h = \sqrt{\alpha^2 + \beta^2}$ is Rayleigh distributed and the phase $\theta = \tan^{-1}\frac{\beta}{\alpha}$ is uniformly distributed between $[0, 2\pi]$.

- h is called the Rayleigh fading coefficient.

Def: Instanteanous SNR:

$$SNR_i = \frac{|h|^2 E[X^2]}{E[\eta^2]} = \rho \cdot |h|^2$$

where $\rho = \frac{E[X^2]}{E[\eta^2]}$ = SNR for AWGN.

Capacity of Flat Fading Rayleigh channels

$$C = E_h\left\{\log_2(1+\rho\cdot|h|^2)\right\}$$

Ergodic (or Average) capacity

Def: Instanteanous capacity

$$C_i = \log_2(1+\rho\cdot|h|^2)$$

Fading and MIMO Capacity

Notice that the instantaneous capacity of fading channels is a random variable which depends on the distribution of the fading coefficient h.

MIMO System Model

$$\vec{Y} = H\vec{X} + \vec{n}$$

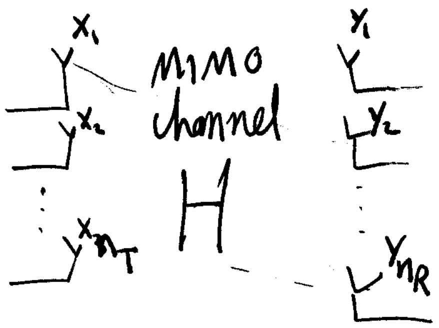
where:

$\vec{X}$ is $n_T \times 1$ transmit vector

$\vec{Y}$ is $n_R \times 1$ receive vector

H is the $(n_R \times n_T)$ channel matrix.

$\vec{n}$ is the $(n_R \times 1)$ AWGN



$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n_T} \\ h_{21} & \ddots & & \\ \vdots & & \ddots & \\ h_{n_R 1} & & & h_{n_R n_T} \end{bmatrix}$$

where each coefficient is

$$h_{ij} = \alpha + j\beta = \sqrt{\alpha^2+\beta^2} \cdot e^{-j\tan^{-1}\frac{\beta}{\alpha}} = |h_{ij}|\, e^{j\phi_{ij}}$$

If α and β are independent and normally distributed random variables

$\Rightarrow |h_{ij}|$ is a Rayleigh distributed random variable.

MIMO Channel Capacity

$$C = E_H \left\{ \max_{P(\vec{X}) : tr(\Phi) \leq P_T} I(x;y) \right\}$$

where $\Phi = E\{XX^H\}$ is the covariance matrix of the transmit signal vector $\vec{X}$. And A^H is the Hermitian transpose. For a given H,

$$I(x;y) = h(y) - h(y|X)$$
$$= h(y) - h(Hx+n|X)$$
$$= h(y) - h(n|X)$$
$$= h(y) - h(n)$$ since n and X are independent.

Theorem: For a complex Gaussian vector $Y \in C^n$, the differential entropy is

$$h(y) \leq \log_2 \det(\pi e K)$$

with equality if y is circularly symmetric complex gaussian with $E\{yy^H\} = K$.

— $E\{yy^H\} = E\{(Hx+n)(Hx+n)^H\}$
$$= E\{Hxx^H H^H\} + E\{nn^H\}$$
$$= H\Phi H^H + K^n$$
$$= K^d + K^n$$

$\therefore$ $I = h(y) - h(n)$
$$= \log_2 [\det(\pi e(K^d + K^n))] - \log_2 [\det(\pi e K^n)]$$
$$= \log_2 [\det(K^d + K^n)] - \log_2 [\det(K^n)]$$
$$= \log_2 \{\det((K^d + K^n)(K^n)^{-1})\}$$
$$= \log_2 \{\det(K^d (K^n)^{-1} + I_{nR})\}$$
$$= \log_2 \{\det(H\Phi H^H (K^n)^{-1} + I_{nR})\}$$

— When the transmitter has no knowledge about the channel, it is optimal to use a uniform power distribution.

$$\Rightarrow \Phi = \frac{P_T}{n_T} I_{nT}$$

— Assuming also uncorrelated noise

$$\Rightarrow K^n = \sigma^2 I_{nR}$$

Fading and MIMO Capacity

$\Rightarrow$ Ergodic capacity for a complex AWGN MIMO channel is:

$$C = E_H\left\{\log_2\left[\det\left(I_{n_R} + \frac{P_T}{\sigma^2 n_T} H H^H\right)\right]\right\}$$

$$C = E_H\left\{\log_2\left[\det\left(I_{n_R} + \frac{\rho}{n_T} H H^H\right)\right]\right\}$$

- Using Eigenvalue Decomposition

$$H H^H = E \Lambda E^H$$

where E is the eigenvector matrix with orthonormal columns and Λ is a diagonal matrix with the eigenvalues.

$$\Rightarrow C = E_H\left\{\log_2\left[\det\left(I_{n_R} + \frac{\rho}{n_T} E \Lambda E^H\right)\right]\right\}$$

$$= E_H\left\{\log_2\left[\det E\left[\det\left(E^H I_{n_R} E + \frac{\rho}{n_T}\Lambda\right)\right]\det E^H\right]\right\}$$

since $\det E = 1$ and $E^H E = I$

$$\Rightarrow C = E_H\left\{\log_2\left[\det\left[\underbrace{I_{n_R} + \frac{\rho}{n_T}\Lambda}_{\text{Diagonal Matrix}}\right]\right]\right\}$$

$\Rightarrow$ ~~$C = E_H\{\log$~~

$$C = E_H\left\{\sum_{i=1}^{k}\log_2\left(1 + \frac{\rho}{n_T}\lambda_i\right)\right\}$$

where $k = \mathrm{rank}(H) \leq \min(n_T, n_R)$

Fading and MIMO Capacity

- Water Filling

The capacity maximization problem can be represented as

$$\max_{\Phi} \log_2\left[\det\left(I_{n_R} + \frac{1}{\sigma^2} H \Phi H^H\right)\right]$$

subject to $tr(\Phi) \leq P$

where:

~~Φ~~ $\Phi = E\{xx^H\}$

P is the power budget

- Using Eigenvalue Decomposition

$$HH^H = E \Lambda E^H$$

where $\Lambda = diag\{\lambda_1, \ldots, \lambda_k\}$

where $k = rank(H) \leq \min(n_T, n_R)$

- Let Φ be a diagonal matrix such that $\Phi = diag\{q_1, q_2, \ldots, q_{n_T}\}$

$\Rightarrow$ The optimal value for q_i is

$$q_i = \left(\nu - \frac{\sigma^2}{\lambda_i}\right)^+$$

where ν is the water level such that $\sum^{n_T} q_i = P$