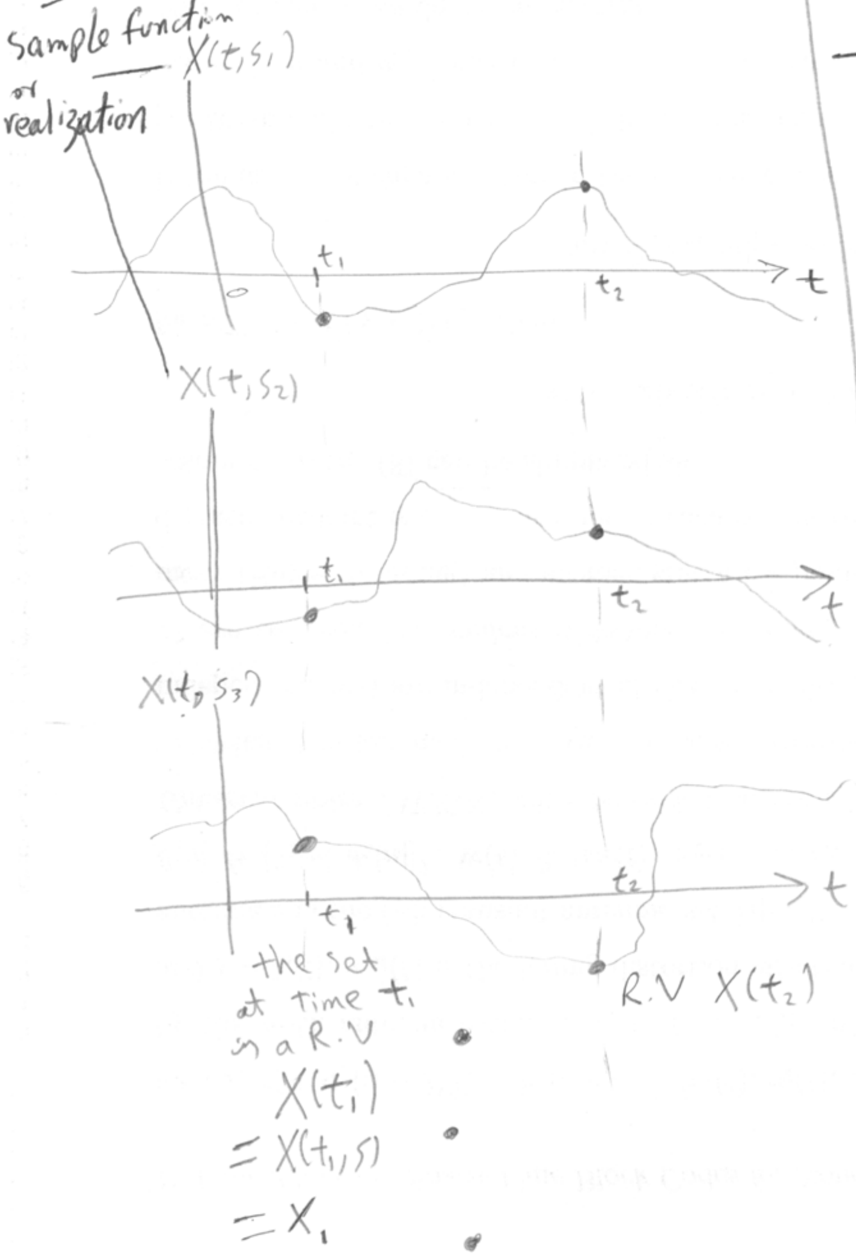
Ch. 6 Random Processes
"Temporal Characteristics"

- Time ~~Random~~ Random Waveforms

example: Bit stream ~~of~~ in Binary Comm. System.

example: Noise and interference.

6.1 The Random Process Concept



Def:

- Ensemble is the set that contains all members of the random process $X(t,s)$

- Sample function or Ensemble Member: is one realization of the random process.

- When t is fixed at t_1 and s is variable, we get a random variable (R.V) $X(t_1)$

Notation:

$X(t) \equiv$ Random Process

$x(t) \equiv$ one realization of a Random Process.

$X(t_i) \equiv$ a R.V. at time t_i

also $X_i \equiv$ a R.V. at time t_i.

$E\{X_i\} \equiv$ ensemble average

6.2 Stationarity and Independence

- A random process becomes a random variable when time is fixed at some particular value.
- The random variable will have statistical properties such as mean, moments and variance.
- At different times, you will get different random variables.

Def: A random process is called stationary if all its statistical properties do not change with time.

Distribution and Density Functions

* First order distribution

Let $X_1 = X(t_1)$ be a R.V taken from a Random Process $X(t)$

then $F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$

* Second order distribution

Let $X_1 = X(t_1)$ and $X_2 = X(t_2)$ be two R.Vs taken from a Random Process $X(t)$, then

$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, ...$

Similarly,

$F_X(x_1, \ldots, x_N; t_1, \ldots, t_N)$

$= P\{X(t_1) \leq x_1, \ldots, X(t_N) \leq x_N\}$

Statistical Independence

Two processes $X(t)$ and $Y(t)$ are statistically independent if the random variable group $X(t_1), X(t_2), \ldots, X(t_N)$ is independent of the group $Y(t_1'), Y(t_2'), \ldots, Y(t_M')$

$\Rightarrow f_{XY}(x_1, \ldots, x_N, y_1, \ldots, y_M; t_1, \ldots, t_N, t_1', \ldots, t_M')$

$= f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) \cdot f_Y(y_1, \ldots, y_M; t_1', \ldots, t_M')$

First-Order Stationary Processes

$$f_X(x_1; t_1) = f_X(x_1; t_1+\Delta)$$

$\Rightarrow$ first-order density function does not change with a time shift.

Thus, the process mean value will be a constant.

$$E[X(t)] = \bar{X} = \text{constant at any } t$$

$$\Rightarrow E[X(t_1)] = E[X(t_2)]$$

<u>Proof</u>: Let $X_1 = X(t_1)$ and $X_2 = X(t_2)$

$$\Rightarrow E[X_1] = E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_1)\, dx_1$$

also
$$E[X_2] = E[X(t_2)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_2)\, dx_1$$

Now, Let $t_2 = t_1 + \Delta$

$\Rightarrow$ for First-order stationary

$$f_X(x_1; t_2) = f_X(x; t_1)$$

$$\Rightarrow E[X_2] = E[X_1]$$

Second-Order Stationary

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta, t_2+\Delta)$$

$\Rightarrow$ Second order moment is a function of the time difference $t_2 - t_1$.

Also, a second-order stationary process is also a first-order stationary

— Autocorrelation function

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

— For a second-order stationary process

$$R_{XX}(t_1, t_1+\tau) = E[X(t_1)X(t_1+\tau)] = R_{XX}(\tau)$$

"The autocorrelation is a function of the time difference"

Wide-Sense Stationary Process

$$E[X(t)] = \bar{X} = \text{constant}$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Example 6-2-1

$$X(t) = A\cos(w_0 t + \theta)$$

where A and w_0 are constants and θ is a random phase uniformly distributed between $[0, 2\pi)$

- The mean is

$$E[X(t)] = \int_0^{2\pi} A\cos(w_0 t + \theta)\frac{1}{2\pi}\,d\theta$$

$$= 0$$

- The Autocorrelation

$$R_{XX}(t, t+\tau) = E\left[\begin{array}{l} A\cos(w_0 t + \theta) \\ \cdot A\cos(w_0 t + w_0\tau + \theta) \end{array}\right]$$

$$= \frac{A^2}{2} E\{\cos(w_0\tau) + \cos(2w_0 t + w_0\tau + 2\theta)\}$$

$$= \frac{A^2}{2}\cos(w_0\tau) + \frac{A^2}{2} E\{\underbrace{\cancel{\cos(2w_0 t + w_0\tau + 2\theta)}}_{\to 0}\}$$

$$\Rightarrow R_{XX}(t, t+\tau) = \frac{A^2}{2}\cos(w_0\tau)$$

since R_{XX} is a function of τ only

$\Rightarrow$ $X(t)$ is wide-sense stationary

Def: Cross-correlation function

Let $X(t)$ and $Y(t)$ be two random processes, the cross-correlation function is

$$R_{XY}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$$

Jointly wide-sense stationary:

Let $X(t)$ and $Y(t)$ be two random processes,

if ① $E\{X(t)\} = \bar{X}$
$E\{Y(t)\} = \bar{Y}$

and ② $R_{XY}(t_1, t_2) = R_{XY}(\tau)$

$\Rightarrow$ $X(t)$ and $Y(t)$ are jointly wide-sense stationary.

* N-Order stationary

A random process is stationary to order N if its Nth order density function is invariant to a time shift.

$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = f_X(x_1, \ldots, x_N; t_1+\Delta, \ldots, t_N+\Delta)$$

* Strict-sense stationary:

A process stationary to all orders is called strict-sense stationary.

Time Averages and Ergodicity

Time Average

$$A[\cdot] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\cdot]\,dt$$

Time Mean

$$\bar{x} = A[x(t)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

$x(t)$ ← Sample Function

Time Autocorrelation function

$$\mathcal{R}_{xx}(\tau) = A[x(t)x(t+\tau)]$$
$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\,dt$$

Note: For each sample function, we get different $\bar{x}$ and $\mathcal{R}_{xx}(\tau)$

$\Rightarrow$ $\bar{x}$ and $\mathcal{R}_{xx}(\tau)$ are random variables

On Average,

$E[\bar{x}] = \bar{X} \equiv$ The mean of the Random Process

$E[\mathcal{R}_{xx}(\tau)] = R_{XX}(\tau)$ $\equiv$ Autocorrelation of the Random Process

Ergodic Processes

If the time averages are equal to the statistical averages, the Process is Ergodic

$$\Longrightarrow \bar{x} = \bar{X}$$
$$\mathcal{R}_{xx}(\tau) = R_{XX}(\tau)$$

Jointly Ergodic

Two random processes are called jointly ergodic if they are:

1- Individually ergodic

2- time cross-correlation function Equals the statistical Cross-correlation function.

$$\mathcal{R}_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)\,dt$$
$$= R_{XY}(\tau)$$

~~Mean-Ergodic Processes~~

6.3 Correlation Functions

~~R_x~~ Autocorrelation:

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$

If $X(t)$ is wide-sense stationary

$$\Rightarrow R_{XX}(\tau) = E[X(\tau)X(t+\tau)]$$

Properties:

1. $|R_{XX}(\tau)| \leq R_{XX}(0) \equiv$ Maximum at zero
2. $R_{XX}(-\tau) = R_{XX}(\tau) \equiv$ Even symmetry
3. $R_{XX}(0) = E[X^2(t)] \equiv$ Power of $X(t)$
4. If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.
5. If $X(t)$ is ergodic, zero-mean and has no periodic component
$$\Rightarrow \lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$
6. If $E[X(t)] = \bar{X} \neq 0$ and $X(t) = \bar{X} + N(t)$ where $N(t)$ is at least wide-sense stationary with no periodic components, and $N(t)$ is zero-mean with $R_{NN}(\tau) \to 0$ as $|\tau| \to \infty$, then
$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$
7. $R_{XX}(\tau)$ cannot have an arbitrary shape.
8. Autocorrelation $\overset{\text{Fourier}}{\underset{\text{Transform}}{\Longleftrightarrow}}$ Power density Spectrum

Example 6.3-1

Assume a stationary ergodic process with no periodic components, the Autocorrelation is

$$R_{XX}(\tau) = 25 + \frac{4}{1+6\tau^2}$$

The mean value:

since $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$

$$\Rightarrow \bar{X}^2 = 25$$

and $\bar{X} = \sqrt{25} = \pm 5$

The variance:

$$\sigma_X^2 = E[X^2(t)] - (E[X(t)])^2$$

since $R_{XX}(0) = E[X^2(t)] = 29$

$$\Rightarrow \sigma_X^2 = 29 - 25 = 4$$

Example 6.3-2

$X(t)$ has $R_{XX}(\tau) = e^{-a|\tau|}$, where $a > 0$

Let $Y(t) = X(t)\cos(\omega_0 t + \Theta)$

this Amplitude modulation and Let Θ be a random phase Uniformly dist. between $(-\pi, \pi)$

$$R_{YY}(t, t+\tau) = E[Y(t)Y(t+\tau)]$$

$$= E\begin{bmatrix} X(t)\,X(t+\tau)\cos(\omega_0 t + \Theta) \\ \cdot\cos(\omega_0 t + \omega_0\tau + \Theta) \end{bmatrix}$$

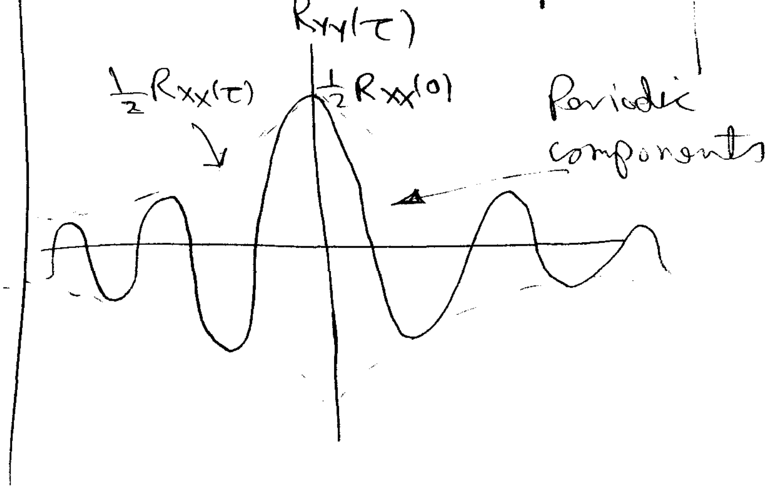
$$= R_{XX}(\tau)\frac{1}{2}E\begin{bmatrix} \cos(\omega_0\tau) \\ +\cos(2\omega_0 t + \omega_0\tau + 2\Theta) \end{bmatrix}$$

since $E[\cos(2\omega_0 t + \omega_0\tau + 2\Theta)] = \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(2\omega_0 t + \omega_0\tau + 2\Theta)\, d\Theta$

$$= 0$$

$$\Rightarrow R_{YY}(t, t+\tau) = \frac{1}{2}R_{XX}(\tau)\cos(\omega_0\tau)$$

$$= R_{YY}(\tau)$$



Cross-Correlation Function

$$R_{xy}(t,t+\tau)=E[X(t)Y(t+\tau)]$$

If $X(t)$ and $Y(t)$ are jointly wide sense stationary

$$\Rightarrow R_{xy}(\tau)=E[X(t)Y(t+\tau)]$$

Orthogonal Processes

$X(t)$ and $Y(t)$ are orthogonal

if $R_{xy}(t,t+\tau)=0$

Statistically Independent Processes

$$R_{xy}(t,t+\tau)=E[X(t)]E[Y(t+\tau)]$$

$=\bar{X}\,\bar{Y}$ if $X(t)$ and $Y(t)$ are wide-sense stationary.

Properties

1. $R_{xy}(-\tau)=R_{yx}(\tau)$
2. $|R_{xy}(\tau)|\le\sqrt{R_{xx}(0)R_{yy}(0)}$
3. $|R_{xy}(\tau)|\le\frac{1}{2}[R_{xx}(0)+R_{yy}(0)]$

See Example 6.3-3 in textbook

Covariance Functions:

Autocovariance Function:

$$C_{xx}(t,t+\tau)$$

$$=E[\{X(t)-E[X(t)]\}\{X(t+\tau)-E[X(t+\tau)]\}]$$

$$=R_{xx}(t,t+\tau)-E[X(t)]E[X(t+\tau)]$$

For ~~Jointly~~ Wide-Sense stationary

$$C_{xx}(\tau)=R_{xx}(\tau)-\bar{X}^2$$

Cross-covariance function:

$$C_{xy}(t,t+\tau)$$

$$=E[\{X(t)-E[X(t)]\}\{Y(t+\tau)-E[Y(t+\tau)]\}]$$

$$=R_{xy}(t,t+\tau)-E[X(t)]E[Y(t+\tau)]$$

For Jointly Wide-Sense stationary

$$C_{xy}(\tau)=R_{xy}(\tau)-\bar{X}\,\bar{Y}$$

Variance $=\sigma_X^2=C_{xx}(0)=R_{xx}(0)-\bar{X}^2$

if $C_{xy}(t,t+\tau)=0$

$\Rightarrow$ $X(t)$ and $Y(t)$ are uncorrelated

$$\Rightarrow R_{xy}(t,t+\tau)=E[X(t)]E[Y(t+\tau)]$$

6.5 Gaussian Random Processes

The Joint Gaussian ~~R.~~ pdf of N random variables taken from the Gaussian Process is

$f_X(x_1, \dots, x_N; t_1, \dots, t_N)$

$$= \frac{\exp\{-(\frac{1}{2})[x-\bar{X}]^t [C_X]^{-1} [x-\bar{x}]\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

where $[x-\bar{X}] = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \\ \vdots \\ x_N - \bar{X}_N \end{bmatrix}$

and $[C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$ = Covariance Matrix

the mean values

$$\bar{X}_i = E[X_i] = E[X(t_i)]$$

$= \bar{X} \equiv$ constant if the process is wide-sense stationary

the covariance matrix

the elements are

$$C_{ik} = C_{X_i X_k}$$

$$= E[(X_i - \bar{X}_i)(X_k - \bar{X}_k)]$$

$$= E[\{X(t_i) - E[X(t_i)]\}\{X(t_k) - E[X(t_k)]\}]$$

$$= C_{XX}(t_i, t_k)$$

~~$= E$~~

$$= R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]$$

For wide sense stationary

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Example 6.5-1

Let $X(t)$ be a wide-sense stationary Random process with mean $\bar{X} = 4$ and Autocorrelation

$$R_{XX}(\tau) = 25e^{-3|\tau|} + 16$$

Find the joint pdf of three random variables defined at

$$t_i = t_0 + \frac{i-1}{2} \quad ; \; i = 1, 2, 3$$

$\Rightarrow t_k - t_i = \frac{k-i}{2}$, i and $k = 1, 2, 3$

$$\Rightarrow R_{XX}(\tau) = R_{XX}(t_k - t_i)$$

$$= 25e^{-3|k-i|/2} + 16$$

Thus, $C_{XX}(t_k - t_i) = 25e^{-3|k-i|/2} + \cancel{16} - 16$

covariance matrix

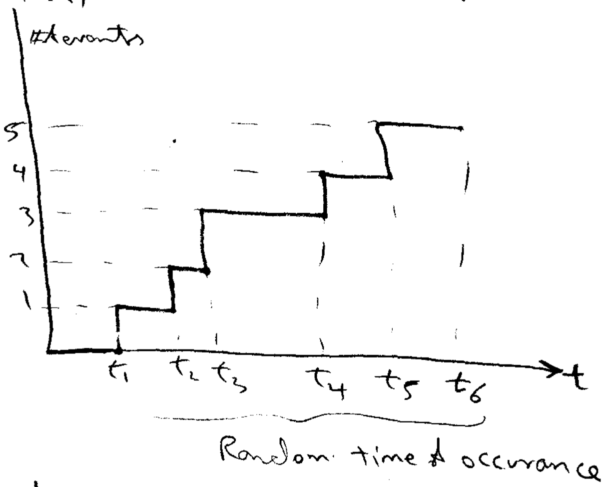
$$[C_X] = \begin{bmatrix} 1 & e^{-3/2} & e^{-6/2} \\ e^{-3/2} & 1 & e^{-3/2} \\ e^{-6/2} & e^{-3/2} & 1 \end{bmatrix} \cdot 25$$

6.6 Poisson Random Process

— It describes the number of times that some event has occurred as a function of time and these events occur at random times.

— "Poisson Counting Process"

$x(t)$ a sample function of $X(t)$



Conditions

1- Only one Event can occur at a time.

2- Occurance times are statistically independent.

⇒ Number of event occurance in any finite interval of time is described by the Poisson distribution.

Let $\lambda \equiv$ the average rate of occurance

Let $b = \lambda t$

⇒ The probability that exactly k occurences over a time interval $(0, t)$ is

$$P\{X(t) = k\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad k = 0, 1, 2, \ldots$$

* the mean = the variance $= \lambda t$

* the power $= E\{X^2(t)\}$

$$= \sigma_X^2 + \bar{X}^2$$

$$= \lambda t [1 + \lambda t]$$