Alghadhban EE345 Ch. 3 P. 1

Ch. 3 Operations on One Random Variable

3.1 Expectation

— Averaging a random variable

Expected value is defined as

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

For discrete pdf;

$$E[X] = \sum_{i=1}^{N} x_i P(x_i)$$

Example

Exponential random variable

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & x > a \\ 0 & x < a \end{cases}$$

$$E[X] = \int_a^{\infty} \frac{x}{b} e^{-\frac{(x-a)}{b}} dx = \frac{e^{a/b}}{b} \int_a^{\infty} x e^{-x/b} dx$$

$$= a + b$$

Note If the pdf is symmetric about a line $x = a$

$$\Rightarrow E[X] = a$$

Expected Value of a Function of a Random Variable.

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

OR

$$E[g(X)] = \sum_{i=1}^{N} g(x_i) P(x_i)$$

Example 3.1-3 (V)

A random voltage can be represented as a Rayleigh R.V with $a = 0$ and $b = 5$.

The power $Y = g(V) = V^2$

Find the average power in V.

$$\text{Power in } V = E[g(V)] = E[V^2]$$

$$= \int_0^{\infty} \frac{2v^3}{5} e^{-v^2/5} dv$$

by letting $\eta = \frac{v^2}{5}$, $d\eta = \frac{2v}{5} dv$

$$\Rightarrow \text{average power in } V = 5 \int_0^{\infty} \eta e^{-\eta} d\eta$$

$$= 5 \text{ W.}$$

Example 3.1-4

Average Information of discrete Source.

Assume a discrete source X with L distinct symbols

$$X = \{x_1, x_2, \cdots, x_L\}$$

the probability of each symbol is

$$P(X) = \{P(x_1), P(x_2), \cdots, P(x_L)\}$$

The information in each symbol is defined as $\log_2\left[\frac{1}{P(x_i)}\right] = -\log_2[P(x_i)]$

$\therefore$ The Average information of the discrete source "Entropy" is

$$E\left[\log_2\left(\frac{1}{P(x_i)}\right)\right] = E\left[-\log_2[P(x_i)]\right]$$

$$= -\sum_{i=1}^{L} P(x_i)\log_2[P(x_i)]$$

Conditional Expectation

$$E[X|B] = \int_{-\infty}^{\infty} x\, f_X(x|B)\,dx$$

Assume that the event B is

$$B = \{X \le b\}$$

$$\Rightarrow f_X(x|X\le b) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx} & x < b \\ 0 & x \geq b \end{cases}$$

Thus

$$E[X|X\le b] = \frac{\int_{-\infty}^{b} x f_X(x)\,dx}{\int_{-\infty}^{b} f_X(x)\,dx}$$

3.2 Moments

Moments about the origin

the n^{th} moment is

$$m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

clearly, $m_0 = 1$

$m_1 = \bar{X}$ = Expected value or the mean

Central Moments

Moments about the mean

$$\mu_n = E[(X-\bar{X})^n] = \int_{-\infty}^{\infty} (x-\bar{X})^n f_X(x)\,dx$$

Note that $\mu_0 = 1$ and $\mu_1 = 0$

— μ_2 is called the variance

— $\mu_2 = \sigma_X^2 = E[(X-\bar{X})^2]$

$$= \int_{-\infty}^{\infty} (x-\bar{X})^2 f_X(x)\,dx$$

— σ_X is called standard deviation

Note that

$$\sigma_X^2 = E[X^2 - 2\bar{X}X + \bar{X}^2]$$
$$= E[X^2] - 2\bar{X}E[X] + \bar{X}^2$$
$$= E[X^2] - \bar{X}^2 = m_2 - m_1^2$$

— μ_3 is called the skew and it is a measure of the asymmetry of the pdf.

$$\mu_3 = E[(X-\bar{X})^3]$$

— If the density function is symmetric about $x = \bar{X}$

$\Rightarrow \mu_n = 0$ for all odd values of n.

skewness $= \dfrac{\mu_3}{\sigma_X^3}$ "Normalized"

Example 3.2-1 and 3.2-2

Chebychev's Inequality

$$P\{|X-\bar{X}| \geq \epsilon\} \leq \frac{\sigma_X^2}{\epsilon^2},$$

for any $\epsilon > 0$.

OR $P\{|X-\bar{X}| < \epsilon\} \geq 1 - \left(\dfrac{\sigma_X^2}{\epsilon^2}\right)$

Markov's Inequality

For ~~any~~ a nonnegative random variable X

$$P\{X \geq a\} \leq E[X]/a \;;\; a > 0$$

3.3 Functions that Give Moments.

— Characteristic Functions

$$\Phi_X(w) = E[e^{jwX}]$$

$$= \underbrace{\int_{-\infty}^{\infty} f_X(x)\, e^{jwx}\, dx}_{\text{Fourier Transform with sign of } w \text{ reversed.}}$$

$\therefore$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_X(w)\, e^{-jwx}\, dw$$

Inverse Fourier Transform with sign of w reversed.

How we get the moments from $\Phi_X(w)$?

$$m_n = (-j)^n \left.\frac{d^n \Phi_X(w)}{dw^n}\right|_{w=0}$$

Note that

$$|\Phi_X(w)| \le \Phi_X(0) = 1$$

Example 3.3-1 For the exponential Random Variable:

$$\Phi_X(w) = \int_a^{\infty} \frac{1}{b} e^{-(x-a)/b} e^{jwx}\, dx$$

$$= \frac{e^{a/b}}{b}\int_a^{\infty} e^{-(\frac{1}{b}-jw)x}\, dx$$

From Appendix C:

$$\Phi_X(w) = \frac{e^{a/b}}{b}\left[\left.\frac{e^{-(\frac{1}{b}-jw)x}}{-(\frac{1}{b}-jw)}\right|_a^{\infty}\right]$$

$$= \frac{e^{jwa}}{1-jwb}$$

The First derivative of $\Phi_X(w)$ is

$$\frac{d\Phi_X(w)}{dw} = e^{jwa}\left[\frac{ja}{1-jwb} + \frac{jb}{(1-jwb)^2}\right]$$

$\therefore$

$$m_1 = (-j)\left.\frac{d\Phi_X(w)}{dw}\right|_{w=0} = a+b$$

— Moment Generating Function

Another function Related to the characteristic function is the moment generating Function

$$M_X(v) = E[e^{vX}]$$

where v is a real number

$$M_X(v) = \int_{-\infty}^{\infty} f_X(x)\, e^{vx}\, dx$$

the moments

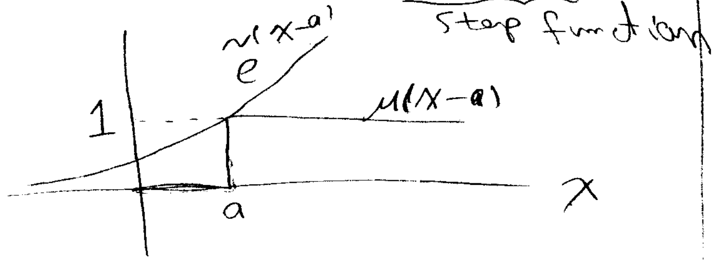
$$m_n = \left.\frac{d^n M_X(v)}{dv^n}\right|_{v=0}$$

— See example 3.3-2

Chernoff's Inequality

Let X be any random variable,
For any real $v > 0$,

$$e^{v(x-a)} \geq \underbrace{u(x-a)}_{\text{step function}}$$



Since

$$P\{X \geq a\} = \int_a^{\infty} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\,u(x-a)\,dx$$

$$\leq \int_{-\infty}^{\infty} f_X(x)\,e^{v(x-a)}\,dx$$

$$\leq e^{-va} M_X(v)$$

$$\therefore \underbrace{P\{X \geq a\} \leq e^{-va} M_X(v)}_{\text{chernoff's inequality}}$$

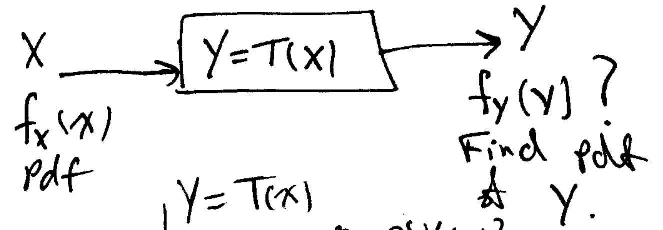
3.4 Transformation of a Random Variable

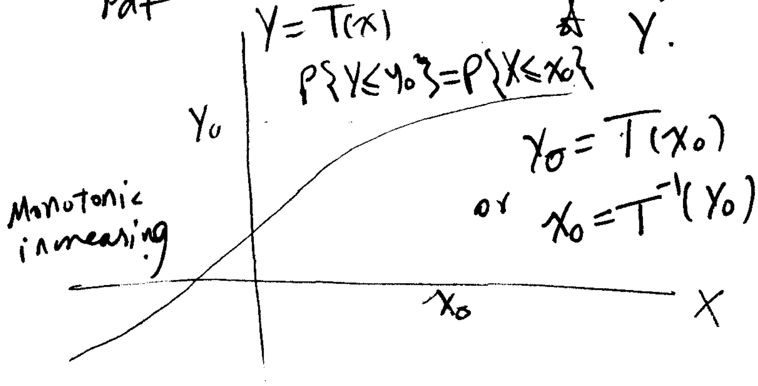
— Monotonic Transformation of a continuous Random Variable.

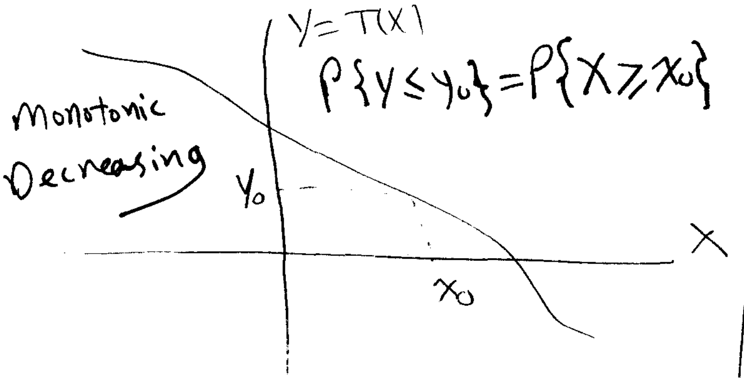
- A transformation T is called monotonically increasing if $T(x_1) < T(x_2)$ for any $x_1 < x_2$.
- It is monotonically increasing if $T(x_1) > T(x_2)$ for any $x_1 < x_2$

(a) Let X be a random variable, the transformation is denoted by

$$Y = T(X)$$







— Since the transformation (T) is one-to-one

$$\Rightarrow P\{Y \le y_0\} = P\{X \le x_0\}$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{-\infty}^{x_0 = T^{-1}(y_0)} f_X(x)\,dx$$

— Differentiate both sides with respect to y_0 using Leibniz's Rule in Appendix G.

$$f_Y(y_0) = f_X[T^{-1}(y_0)] \frac{dT^{-1}(y_0)}{dy_0}$$

— Since the Result applies for all y_0

$$f_Y(y) = f_X[T^{-1}(y)] \frac{dT^{-1}(y)}{dy}$$

$$f_Y(y) = f_X(x) \frac{dx}{dy}$$

— For monotonically decreasing transformation

$$P\{Y \le y_0\} = P\{X \ge x_0\}$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = 1 - \int_{-\infty}^{x_0} f_X(x)\,dx$$

— Repeating the steps, we get a negative right hand side. However, since the slope $\frac{dx}{dy}$ will be negative also

$\Rightarrow$ In General,

$$\boxed{f_Y(y) = f_X(T^{-1}(y)) \left| \frac{dT^{-1}(y)}{dy} \right|}$$

$$\boxed{f_Y(y) = f_X(T^{-1}(y))\left|\frac{dT^{-1}(y)}{dy}\right|} \quad \text{or} \quad \boxed{f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|}$$

Example

Let $Y = aX + b$

$$\Rightarrow \quad X = \frac{Y-b}{a}$$

$$\frac{dx}{dy} = \frac{1}{a}$$

$$\therefore \quad f_Y(y) = f_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right|$$

Let X be a gaussian R.V

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$$

$\therefore$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{\left[\frac{y-b}{a} - a_X\right]^2}{2\sigma_X^2}}\left|\frac{1}{a}\right|$$

$$= \frac{1}{\sqrt{2\pi a^2\sigma_X^2}} e^{-\frac{[y-(aa_X+b)]^2}{2a^2\sigma_X^2}}$$

Let $a_Y = a a_X + b$

and $\sigma_Y^2 = a^2\sigma_X^2$

$\Rightarrow$ Y is also gaussian R.V

- linear transformation of a gaussian

Leibniz's Rule

Appendix G

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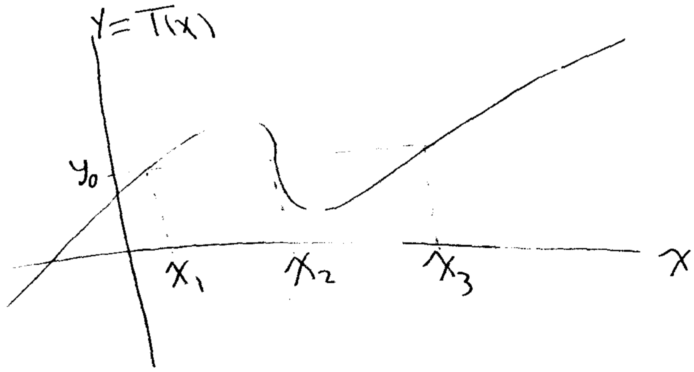
$$G(u) = \int_{\alpha(u)}^{\beta(u)} H(x,u)\,dx$$

$$\Rightarrow$$

$$\frac{dG(u)}{du} = H[\beta(u),u]\frac{d\beta(u)}{du} - H[\alpha(u),u]\frac{d\alpha(u)}{du} + \int_{\alpha(u)}^{\beta(u)} \frac{\partial H(x,u)}{\partial u}\,dx$$

- Nonmonotonic Transformation of a Continuous Random Variable.



* not one-to-one Mapping

Note that the event $\{Y \le y_0\}$ corresponds to the event

$\equiv \{X \le x_1$ and $x_2 \le X \le x_3\}$

$\equiv$ {x values yielding $Y \le y_0$}

$\equiv \{x \mid Y \le y_0\}$

- The Method to find $f_Y(y)$ is

$$f_Y(y_0) = \frac{d}{dy_0}\int_{\{x \mid Y \le y_0\}} f_X(x)\,dx$$

$$f_Y(y) = \sum_n \frac{f_X(x_n)}{\left|\frac{dT(x)}{dx}\right|_{x=x_n}}$$

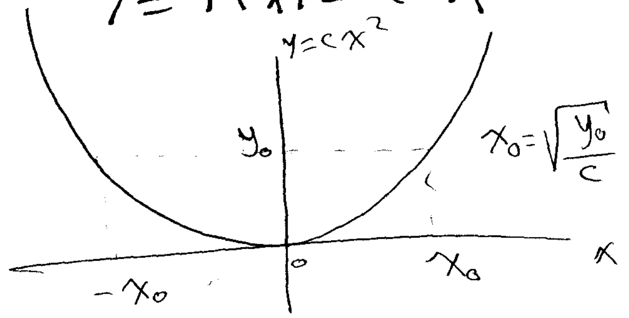
where the sum is taken over n, which is the number of roots of the equation

$$y = T(x)$$

with roots x_n, $n = 1, 2, \ldots$

Example 3.4-2

$$Y = T(X) = cX^2$$



From the above graph, the event $\{Y \le y\} \equiv \left\{-\sqrt{\frac{y}{c}} \le X \le \sqrt{\frac{y}{c}}\right\}$

$$\therefore f_Y(y) = \frac{d}{dy}\int_{-\sqrt{\frac{y}{c}}}^{\sqrt{\frac{y}{c}}} f_X(x)\,dx \quad , y \geq 0$$

Using Leibniz's Rule

$$f_Y(y) = f_X\left(\sqrt{\tfrac{y}{c}}\right)\frac{d\left(\sqrt{\frac{y}{c}}\right)}{dy} - f_X\left(-\sqrt{\tfrac{y}{c}}\right)\frac{d\left(-\sqrt{y/c}\right)}{dy} + \int_{-\sqrt{y/c}}^{\sqrt{y/c}} \underbrace{\frac{\partial f_X(x)}{\partial y}}_{0}\,dx$$

$$\therefore f_Y(y) = f_X\left(\sqrt{\tfrac{y}{c}}\right)\left(\frac{1}{2\sqrt{cy}}\right) - f_X\left(-\sqrt{\tfrac{y}{c}}\right)\left(\frac{-1}{2\sqrt{cy}}\right)$$

$$= \frac{f_X\left(\sqrt{\frac{y}{c}}\right) + f_X\left(-\sqrt{\frac{y}{c}}\right)}{2\sqrt{cy}} \quad y \geq 0$$

Another Method, use

$$f_Y(y) = \frac{f_X(x_1)}{\left|\frac{dT(x)}{dx}\right|_{x=x_1}} + \frac{f_X(x_2)}{\left|\frac{dT(x)}{dx}\right|_{x=x_2}}$$

the roots are: $x_1 = -\sqrt{\frac{y}{c}}$

$x_2 = \sqrt{\frac{y}{c}}$

$T(x) = cx^2$

$\frac{dT(x)}{dx} = 2cx$

$\therefore$ at $x_1 \Rightarrow \left.\frac{dT(x)}{dx}\right|_{x=x_1} = -2\sqrt{cy}$

at $x_2 \Rightarrow \left.\frac{d(T(x))}{dx}\right|_{x=x_2} = 2\sqrt{cy}$

$\therefore$

$$f_Y(y) = \frac{f_X(\sqrt{\frac{y}{c}}) + f_X(-\sqrt{\frac{y}{c}})}{2\sqrt{cy}} \; ; \; y > 0$$

Transformation of a Discrete Random Variable:

$$f_X(x) = \sum_n P(x_n)\delta(x - x_n)$$

$$F_X(x) = \sum_n P(x_n)u(x - x_n)$$

Let $Y = T(X)$,

case 1 monotonic transformation

$\Rightarrow$ one-to-one mapping

$\Rightarrow P(y_n) = P(x_n)$

where $y_n = T(x_n)$

Thus, $f_Y(y) = \sum_n P(y_n)\delta(y - y_n)$

$F_Y(y) = \sum_n P(y_n) \cdot u(y - y_n)$

case 2 not-monotonic

$P(y_n) = \sum P(x_n)$

for which $y_n = T(x_n)$

Example 3.4-3 Discrete transformation

Let $X = \{-1, 0, 1, 2\}$

$P(X) = \{0.1, 0.3, 0.4, 0.2\}$

$\Rightarrow f_X(x) = 0.1\delta(x+1) + 0.3\delta(x) + 0.4\delta(x-1) + 0.2\delta(x-2)$

Let $Y = 2 - X^2 + \left(\frac{X^3}{3}\right)$

$\Rightarrow Y = \left\{\frac{2}{3}, 2, \frac{4}{3}\right\}$

so

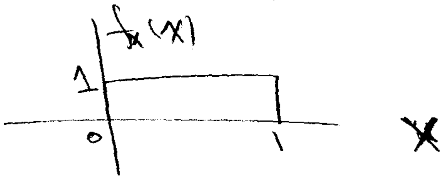
$P(Y) = \{0.1 + 0.2, 0.3, 0.4\}$

$= \{0.3, 0.3, 0.4\}$

$f_Y(y) = 0.3\delta\{y - \frac{2}{3}\} + 0.4\delta\{y - \frac{4}{3}\} + 0.3\delta(y-2)$

3.5 Computer Generation of One Random Variable.

Assume that we have a function that can generate a uniform Random Variable from (0, 1)



Then, How can we transform this Uniform R.V to other distributions?

Let $f_X(x)$ be the pdf of a uniform R.V. and $F_X(x)$ be the CDF.

Let $y = T(X)$ be a transformation of X

then, we saw that for monotonically nondecreasing transformation

$$\Rightarrow F_Y\{y = T(x)\} = F_X(x)$$

$$= \underbrace{x}_{\text{uniform CDF}}, \quad \text{for } 0 < x < 1$$

$$\Rightarrow F_Y^{-1}\{F_Y(y = T(x))\} = F_Y^{-1}(x)$$

$$\Rightarrow Y = F_Y^{-1}(x)$$

$$\boxed{Y = T(x) = F_Y^{-1}(x)}$$

— **Method** (1) Generate uniform R.V X

(2) Let $F_Y(y)$ be the CDF of the required distribution

(3) use the following transformation

$$Y = T(x) = F_Y^{-1}(x)$$

Example 3.5-1

Let Y be a Rayleigh pdf with $a=0$

$$\Rightarrow F_Y(y) = 1 - e^{-y^2/b}$$

Let $F_Y(y) = x$ and solve for y

$$1 - e^{-y^2/b} = x$$

$$e^{-y^2/b} = 1 - x$$

$$\frac{-y^2}{b} = \ln(1-x)$$

$$\Rightarrow y = \sqrt{-b\ln(1-x)}$$

for $0 < x < 1$