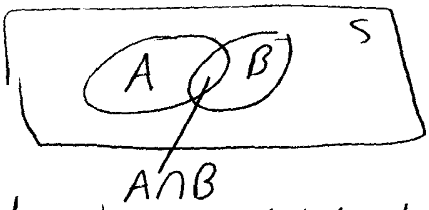
1.4 Joint and Conditional Prob.

Joint Probability



A and B are not mutually exclusive.

what is the probability of the Joint event $A\cap B$?

From the above Venn diagram,

Joint Prob. $\rightarrow P(A\cap B) = P(A) + P(B) - P(A\cup B)$

Thus,
$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$
$$\leq P(A) + P(B)$$

Equality holds only for mutually exclusive events. i.e "$A\cap B = \emptyset$"

Conditional Probability

Conditional probability of an event A, given B is

Joint Prob.

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \qquad P(B) > 0$$

"Prob. of event A may depend on event B"

- If A and B are mutually exclusive
$\Rightarrow P(A|B) = 0$ because $A\cap B = \emptyset$

Total Probability

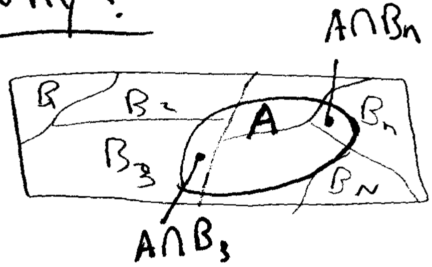
- The total Probability of event A defined on a sample space S can be expressed as:

$$P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n)$$

where $B_m \cap B_n = \emptyset$ for $m \neq n = 1, 2, \cdots, N$

and $\bigcup_{n=1}^{N} B_n = S$

Why?



By Inspection
$$P(A) = \sum_{n=1}^{N} P(A\cap B_n)$$

From conditional Prob. Def.:
$$P(A\cap B_n) = P(A|B_n)P(B_n)$$

Thus,
$$P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n)$$

Bayes' Theorem

Since $P(B_n|A) = \dfrac{P(B_n \cap A)}{P(A)}$

and $P(A/B_n) = \dfrac{P(A \cap B_n)}{P(B_n)}$

Bayes Theorem $\Longrightarrow P(B_n|A) = \dfrac{P(A|B_n)\,P(B_n)}{P(A)}$

and also

$$P(B_n|A) = \frac{P(A|B_n)\,P(B_n)}{\sum_{m=1}^{N} P(A|B_m)\,P(B_m)}$$

* Solve Example 1.4-2 page 18

1.5 Independent Events

- Let two events A and B have nonzero probabilities

$$P(A) \neq 0 \text{ and } P(B) \neq 0$$

- These two events are statistically independent if the probability of occurance of one event is not affected by the occurence of the other event.

$$\Rightarrow P(A/B) = P(A)$$
$$\text{and } P(B/A) = P(B)$$

since $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

- For independent events

$$\Longrightarrow P(A \cap B) = P(A)\,P(B)$$

$\Rightarrow$ Joint Prob. of A and B ~~must~~ equal to the product of the two event Prob.

- Recall that if A and B are mutually exclusive

$$\Rightarrow P(A \cap B) = 0$$

$\Longrightarrow$ A and B can't be both mutually exclusive and statistically independent.

$\Rightarrow A \cap B \neq \phi$ "must have"

1.6 Combined Experiment

- A combined experiments consist of forming a single experiment by suitably combining individual experiments called subexperiments.

- Combined Sample Space

~~Let S be~~

- Two subexperiments with sample spaces S_1 and S_2

- The combined sample space is

$$S = S_1 \times S_2$$

and $(s_1, s_2) \in S$

if S_1 has M elements and S_2 has N elements

$\Rightarrow$ S has MN elements

- Solve example 1.6-1

- Events on the Combined Space

Let S_1 and S_2 be the sample spaces of two subexperiments then,

$$C = A \times B$$

is an event defined on S consisting of all pairs (s_1, s_2) such that $s_1 \in A$ and $s_2 \in B$

- Probabilities on Combined Experiments.

For N independent experiments,
$P(A_1 \times A_2 \times \cdots \times A_N) = P(A_1)P(A_2)\cdots P(A_N)$

— See Example 1.6-5

- Permutations
Orderings of r elements taken from n elements in

$$= n(n-1)(n-2)\cdots(n-r+1)$$
$$= \frac{n!}{(n-r)!} = P^n_r$$

ordering is important in permutations.

— Note that
$0! = 1$
$\binom{n}{0} = 1$
and $\binom{n}{n} = 1$
also $\binom{n}{r} = \binom{n}{n-r}$

— Combinations
When the order of elements in a sequence is not important, the number of permutations is reduced by $r!$.

- The number of combinations of r things taken from n things is
$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

- $\binom{n}{r}$ is called binomial coefficients because

$\ldots n \sum^{n} \binom{n}{} x^r y^{n-r}$

1.7 Bernoulli Trials

Let S have two elements only

$S = \{A, \bar{A}\}$

If we repeat the experiment N times and determine the probability that A is observed exactly k times out of the N trials

$\Longrightarrow$ Bernoulli Trials

Let $P(A) = p$

and $P(\bar{A}) = 1 - p$

$\therefore$

$P(A$ occuring k times for one sequence$)$

$$= \underbrace{P(A)P(A)\cdots P(A)}_{k}\ \underbrace{P(\bar{A})P(\bar{A})\cdots P(\bar{A})}_{N-k}$$

$$= p^k (1-p)^{N-k}$$

Since there are $\binom{N}{k}$ possible combinations

$\Rightarrow P\{A$ occurs exactly k times$\}$

$$= \binom{N}{k} p^k (1-p)^{N-k}$$

when N, k, and $(N-k)$ are large, we can use the following approximations

– De Moivre–Laplace approximation

$$\binom{N}{k} p^k (1-p)^{N-k} \simeq \frac{1}{\sqrt{2\pi N p(1-p)}} \exp\left[-\frac{(k-Np)^2}{2Np(1-p)}\right]$$

– Poisson Approximation

$$\binom{N}{k} p^k (1-p)^{N-k} \simeq \frac{(Np)^k e^{-Np}}{k!}$$