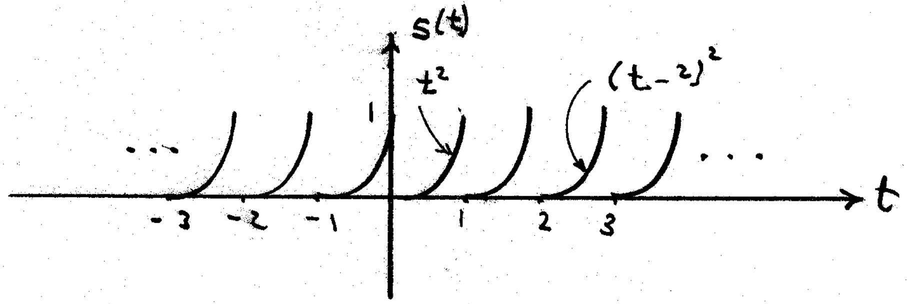
EE 207 Problem Session I

a)



Trigonometric form of Fourier Series

$$s(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) \right]$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} s(t)\, dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} s(t) \cos(2\pi n f_0 t)\, dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} s(t) \sin(2\pi n f_0 t)\, dt$$

$$T = 1 \quad , \quad f_0 = 1/T = 1$$

$$a_0 = \int_0^1 t^2\, dt = \frac{1}{3} t^3 \Big|_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$$

$$a_n = 2\int_0^1 t^2 \cos(2\pi n t)\, dt$$

interation by parts, let

$u = t^2$ $\qquad$ $dv = \cos(2\pi n t)\, dt$

$du = 2t\, dt$ $\qquad$ $v = \frac{1}{2\pi n} \sin(2\pi n t)$

$$a_n = 2\left[\cancelto{0}{\frac{t}{2\pi n} \sin(2\pi n t)]\Big|_0^1} - \int_0^1 \frac{2t}{2\pi n} \sin(2\pi n t)\, dt \right.$$

let $u' = t$ $\qquad$ $dv' = \sin(2\pi n t)\, dt$

$du' = dt$ $\qquad$ $v' = \frac{-1}{2\pi n} \cos(2\pi n t)$

$$a_n = \frac{-1}{\pi n} \left[\frac{-t}{2\pi n} \cos(2\pi n t)\Big|_0^1 + \int_0^1 \frac{1}{2\pi n} \cos(2\pi n t)\, dt \right]$$

$$= \frac{+1}{\pi n} \left[\frac{1}{2\pi n} \cos(2\pi n) - 0 - \left(\frac{1}{2\pi n}\right)^2 \cancelto{0}{\sin(2\pi n)\Big|_0^1} \right]$$

$$\boxed{a_n = \frac{(-1)^{n+2}}{2(\pi n)^2}}$$

$$b_n = 2\int_0^1 t^2 \sin(2\pi n t)\, dt$$

integration by parts, let

$u = t^2$ $\qquad dv = \sin(2\pi n t)\, dt$

$du = 2t\, dt$ $\qquad v = -\frac{1}{2\pi n}\cos(2\pi n t)$

$$b_n = 2\left[\frac{-t^2}{2\pi n}\cos(2\pi n t)\Big|_0^1 + \int_0^1 \frac{2t}{2\pi n}\cos(2\pi n t)\, dt \right]$$

$$= 2\left[\frac{-1}{2\pi n}\cos\overset{1}{\cancel{(2\pi n)}} + 0 + \int_0^1 \frac{t}{\pi n}\cos(2\pi n t)\, dt \right]$$

let $u' = t$ $\qquad dv' = \cos(2\pi n t)\, dt$

$du' = dt$ $\qquad v' = \frac{1}{2\pi n}\sin(2\pi n t)$

$$b_n = 2\left[\frac{(-\cos(2\pi n))}{2\pi n} + \frac{1}{\pi n}\left(\frac{t}{2\pi n}\overset{0}{\cancel{\sin(2\pi n t)}} \right)\Big|_0^1 + \frac{1}{4\pi^2 n^2}\cos(2\pi n t)\Big|_0^1 \right]$$

$$b_n = 2\left[\frac{-\cos(2\pi n)}{2\pi n} + \frac{\cos(2\pi n)}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2} \right]$$

$$b_n = \frac{-1}{\pi n} + \cancel{\frac{1}{2\pi^2 n^2}} - \cancel{\frac{1}{2\pi^2 n^2}}$$

$$\boxed{b_n = -1/\pi n}$$

Exponential Form of Fourier Series

$$s(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} s(t)\, e^{-j2\pi n f_0 t}\, dt$$

$T = 1, \qquad f_0 = 1$

$$C_n = \int_0^1 t^2 e^{-j2\pi n t}\, dt$$

integration by parts, let

$$u = t^2 \qquad dv = e^{-j2\pi n t}\, dt$$

$$du = 2t\, dt \qquad v = \frac{-1}{2\pi n} e^{-j2\pi n t}$$

$$C_n = \frac{-t^2}{2\pi n} e^{-j2\pi n t}\Big|_0^1 + \int_0^1 \frac{t}{\pi n} e^{-j2\pi n t}\, dt$$

$$= \frac{-e^{-j2\pi n}}{2\pi n} + \int_0^1 \frac{t}{\pi n} e^{-j2\pi n t}\, dt$$

$$\text{Let } u' = t \qquad dv' = e^{-j2\pi n t}\, dt$$

$$du' = dt \qquad v' = \frac{-1}{2\pi n} e^{-j2\pi n t}$$

$$C_n = \frac{-1}{\pi n}\left[\cancelto{1}{\cos 2\pi n} - j \cancelto{0}{\sin(-2\pi n)}\right]$$

$$= \frac{-t}{2\pi n} e^{-j2\pi n t}\Big|_0^1 + \int_0^1 \frac{1}{2\pi n} e^{-j2\pi n t}\, dt$$

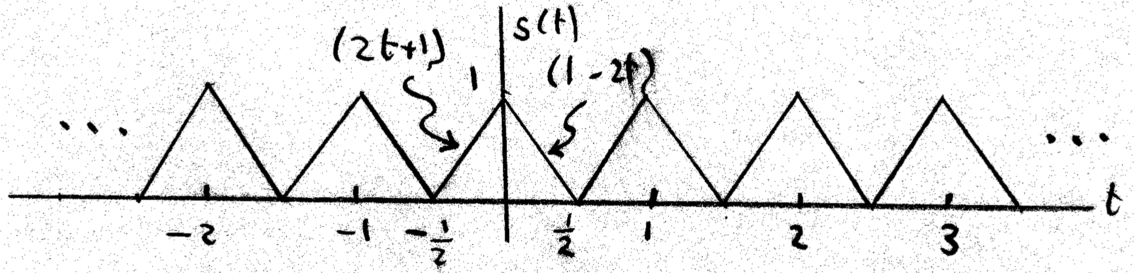
$$= \frac{-e^{-j2\pi n}}{2\pi n} - \frac{1}{4\pi^2 n^2} e^{-j2\pi n}\Big|_0^1$$

$$C_n = \frac{-1}{2\pi n} - \frac{1}{4\pi^2 n^2}, \quad n \neq 0$$

$$C_0 = a_0 = 1/3$$

$$\boxed{\begin{aligned} C_n &= \frac{-1}{2\pi n} - \frac{1}{4\pi^2 n^2}, \quad n \neq 0 \\ C_0 &= 1/3 \end{aligned}}$$

b)



This is an even function $\Rightarrow b_n = 0$

$T = 1 \quad , \quad f_0 = 1$

$$a_0 = \int_{-1/2}^{0} (2t+1)\, dt + \int_{0}^{1/2} (1-2t)\, dt$$

$$= (t^2 + t)\Big|_{-1/2}^{0} + (t - t^2)\Big|_{0}^{1/2}$$

$$= 0 - \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 0$$

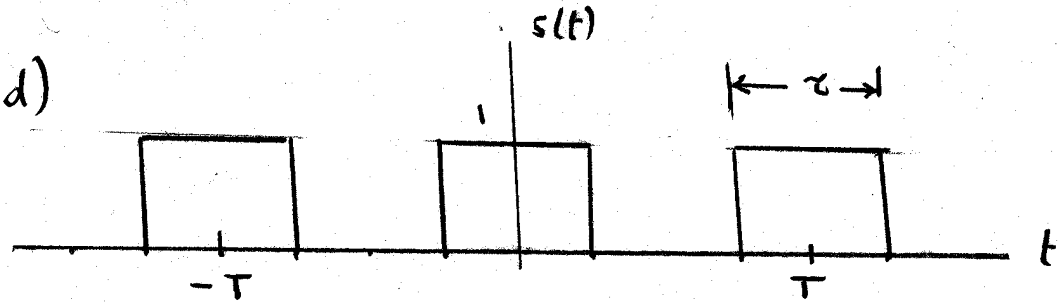
$$\boxed{a_0 = 1/2}$$

$$a_n = 2\int_{-1/2}^{0} (2t+1)\cos(2\pi n t)\, dt + 2\int_{0}^{1/2} (1-2t)\cos(2\pi n t)\, dt$$

$$a_n = \begin{cases} 0 & , \quad n \text{ even} \\ \frac{2}{\pi^2 n^2} & , \quad n \text{ odd} \end{cases}$$

$$\boxed{s(t) = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi^2 n^2} \cos 2\pi n t}$$

d)



This is an even function $b_n = 0$

$$a_0 = \frac{1}{T}\int_{-\tau/2}^{\tau/2} dt = t\Big|_{-\tau/2}^{\tau/2} = \tau/T$$

$$a_n = \frac{2}{T}\int_{-\tau/2}^{\tau/2} \cos\left(\frac{2\pi n t}{T}\right) dt$$

$$= \frac{2T}{2\pi n T}\sin\left(\frac{2\pi n t}{T}\right)\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{1}{\pi n}\left[\sin\left(\frac{\pi n}{T}\tau\right) - \sin\left(\frac{-\pi n \tau}{T}\right)\right]$$

$$= \frac{1}{\pi n}\left[\sin\left(\frac{\pi n}{T}\tau\right) + \sin\left(\frac{\pi n}{T}\tau\right)\right]$$

$$a_n = \frac{2}{\pi n}\sin\left(\frac{\pi n \tau}{T}\right)$$

$$\text{Sinc } x = \frac{\sin x}{x}$$

$$a_n = \frac{2\tau}{\pi\tau}\left(\frac{1}{\frac{\pi n \tau}{T}}\right)\sin\left(\frac{\pi n \tau}{T}\right), \quad \tau \neq 0$$

$$a_n = \frac{2\tau}{T}\,\text{sinc}\left(\pi n\tau/T\right)$$

$$\boxed{s(t) = \frac{\tau}{T} + \sum_{n=1}^{\infty}\frac{4\tau}{T}\,\text{sinc}\left(\frac{\pi n \tau}{T}\right)\cos\left(\frac{2\pi n t}{T}\right)}$$

$$c_n = a_n = \frac{2\tau}{T}\,\text{sinc}\left(\frac{n\pi\tau}{T}\right)$$

$$s(t) = \sum_{n=-\infty}^{\infty}\frac{2\tau}{T}\,\text{sinc}\left(\frac{n\tau}{T}\right)e^{j\frac{2\pi n t}{T}}$$