King Fahd University of Petroleum and Minerals

Department of Electrical Engineering
EE 570
Probability and Stochastic Processes (082)

Midterm Exam
Tuesday Nov. 21, 2011

6:30 PM—8:45 PM

Name ______________________________

ID ______________________________

Problem	Score	Out of
Problem 1)		10
Problem 2)		20
Problem 3)		25
Problem 4)		20
Problem 5)		25
Total		100

Q1: True and False Indicate which of the following statements is true and which is false. There is a penalty of (-1) for every wrong answer.

1. Zero mean uncorrelated random variables are orthogonal.

   True

2. Let X and Y be jointly Gaussian random variables. If X and Y are uncorrelated, then they are independent.

   True

3. A random variable X can not take negative values.

   False

4. For any random variable X, and any function $g(x)$, we have $E[g(X)^2] \geq E[g(X)]^2$.

   True

5. The characteristic function of a random variable completely characterizes the random variable.

   True

6. A linear combination of two Gaussian random variables is Gaussian.

   False

7. Let B an event such that $P(B)$ and $P(\bar{B})$ are both nonzero. Then,

$$P(A|B) + P(A|\bar{B}) = 1$$

   False

8. The events A and $\bar{A}$ are independent.

   False

9. The CDF is an increasing function.

   True

Q2

8

a) Mahmood takes a bus to his work. A new bus runs on Sunday, Tuesday, Thursday, and Saturday while an older bus runs on the other three days. The new bus has a probability of being on time of 2/3 while the older bus has a probability of only 1/3 to be on time. If mahmood arrived late on a given day, what is probability that he took the new bus? L = late ; N_b = New bus ; O_b = Old bus

By Bayes Rule,

$$P(N_b|L) = \frac{P(L|N_b)\,P(N_b)}{P(L)} \quad 2$$

By total Prob: $P(L) = P(L|N_b)\,P(N_b) + P(L|O_b)\,P(O_b)$ 2

$= 1/3 \cdot 4/7 + 2/3 \cdot 3/7 = \frac{4+6}{21} = 10/21$ 2

$$\Rightarrow P(N_b|L) = \frac{1/3 \cdot 4/7}{10/21} = \frac{\frac{4}{21}}{\frac{10}{21}} = 4/10 = \frac{2}{5} \quad 2$$

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b) Let X be a random variable with characteristic function

$$\Phi_X(j\omega) = \alpha\left(\frac{1}{1-j\omega}\right)^3$$

Find α for $\Phi_X(j\omega)$ to be a valid characteristic function

Note that $\Phi_X(j\omega) = E[e^{jX\omega}] \Rightarrow \Phi_X(0) = E[1] = 1$ 2

So, to find α, we need to have $\Phi_X(j\omega)\big|_{\omega=0} = \alpha\left(\frac{1}{1-j\omega}\right)^3\Big|_{\omega=0} = \alpha = 1$ 1

$\Rightarrow \boxed{\alpha = 1}$ 2

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c) Let X be a random variable with pdf $f(x) = \exp(-x)u(x)$ and define the random variable Y as

$$Y = \begin{cases} 2X & \text{for } 0 \le X < 1 \\ 2 & \text{for } 1 \le X \end{cases}$$

find and sketch the pdf of Y.

For $y < 2$; $\Pr\{Y \le y\} = \Pr\{2x \le y\} = \Pr\{x \le y/2\} = \int_0^{y/2} \exp(-x)\,dx$

$F(y) = 1 - \exp(-y/2)$ 3

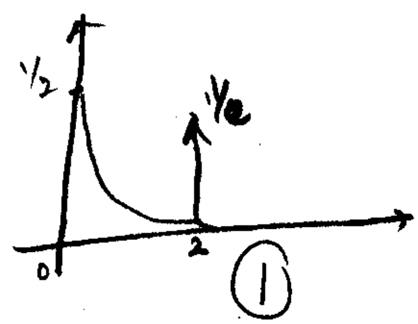
$f(y) = 1/2 \exp(-y/2)$

For $y = 2$; $\Pr\{Y = 2\} = \Pr\{1 \le x\} = \int_1^{\infty} \exp(-x)$

$F(y) = 1/e$

$f(y) = \frac{1}{e}\delta(y-2)$ 2

$f(y) = 0$ otherwise 1



1

Q3 Let X be a random variable with pdf

$$f(x) = \frac{1}{4}\frac{1}{\sqrt{4\pi}}\exp\left(-\frac{(x-1)^2}{4}\right) + \alpha\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+1)^2}{2}\right) \quad (*)$$

$\boxed{6}$

a) Find α for $f(x)$ to be a valid pdf.

Integrate both sides of $(*)$ to get

$$\frac{1}{4}\int\frac{1}{\sqrt{4\pi}}\exp\left(-\frac{(x-1)^2}{4}\right)dx + \alpha\int\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x+1)^2}{2}\right)dx = 1$$

1 ... 1 ... (2)

$$\Rightarrow \frac{1}{4} + \alpha = 1 \Rightarrow \boxed{\alpha = 3/4}$$ (2)

$\boxed{7}$

b) For the remaining part of the problem, assume that the correct value of α to be $\alpha = \frac{1}{4}$. Find the mean and variance of X.

$$E[X] = \int_{-\infty}^{\infty} x f(x) = \frac{1}{4}\int\frac{x}{\sqrt{4\pi}}\exp\left(-\frac{(x-1)^2}{4}\right) + \frac{1}{4}\int_{-\infty}^{\infty}\frac{x}{\sqrt{2\pi}}\exp\left(-\frac{(x+1)^2}{2}\right)dx$$

(1) 1 (1) (1) -1

$$= \frac{1}{4}\times 1 + \frac{1}{4}\times -1 = 0$$

$$\sigma^2 = E[X^2] \overset{(1)}{=} \frac{1}{4}\int\frac{x^2}{\sqrt{4\pi}}\exp\left(-\frac{(x-1)^2}{4}\right) + \frac{1}{4}\int\frac{x^2}{\sqrt{2\pi}}\exp\left(-\frac{(x+1)^2}{2}\right)dx$$

$$1/4 \times (2+(1)^2) \text{ (1)} \qquad \frac{1}{4}\times (1+(-1)^2) \text{ (1)}$$

$$= 5/4 \text{ (1)}$$

$\boxed{7}$

c) Find the probability $P\{-1 < X \le 2\}$. Express your answer in terms of the Q function.

$$P\{-1 < X \le 2\} = \int_{-1}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{2}\frac{1}{\sqrt{4\pi}}\exp\left(\frac{(x-1)^2}{4}\right) + \frac{1}{4}\int_{-1}^{2}\frac{1}{\sqrt{2\pi}}\exp\left(\frac{(x+1)^2}{2}\right)$$

(1) (1) (1) (1)

$$= \frac{1}{4}\left(Q\left(\frac{-1-1}{\sqrt{2}}\right) - Q\left(\frac{2-1}{\sqrt{2}}\right) + Q\left(\frac{-1+1}{1}\right) - Q\left(\frac{2+1}{1}\right)\right)$$

$$= \frac{1}{4}\left(1 - Q\left(\frac{2}{\sqrt{2}}\right) - Q\left(\frac{1}{\sqrt{2}}\right) + 1/2 - Q(3)\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} - Q(\sqrt{2}) - Q(1/\sqrt{2}) - Q(3)\right) \text{ (2)}$$

$\boxed{5}$

d) By inspecting the pdf of X, Could you say that X is obtained by adding two independent Gaussian random variables? Briefly justify your answer.

No.

The addition to two indep. Gaussian variables gives a Gaussian pdf.

The pdf above is not that of a Gaussian pdf

Q4 Let X and Y two random variables with joint pdf

$$f(x,y) = \begin{cases} 2 & \text{for } 0 < x < 1, \ \& \ 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

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a) Find the marginal densities $f(x)$ and $f(y)$.

$f(x) = \int_0^x f(x,y)\,dy = 2x \quad (0 < x < 1)$

$f(y) = \int_y^1 f(x,y)\,dx = 2 - 2y \quad (0 < y < 1)$

$2 - 2y \Big|_0^1 = 1$

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b) Find the best estimate of X in the mean square sense.

$E[x] = \int_0^1 x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3} \Rightarrow \boxed{E[X] = \frac{2}{3}}$

$E(x)$ is the ~~worse~~ best estimate in the mean square sense when we have no observation.

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c) Find the best estimate of X in the mean square sense given that $Y = \frac{1}{2}$.

Best estimate will be the MMSE estimate of X given $Y = \frac{1}{2}$.

$E[X/Y = \frac{1}{2}] = \int_0^1 x f(x|y=\tfrac{1}{2})\,dx$

$= \int_0^1 x \frac{f(x,y)}{f(y)}\Big|_{y=\frac{1}{2}} dx$

$= \int_{1/2}^1 2x\,dx = x^2\Big|_{1/2}^1 = 1 - 1/4 = 3/4$

~~$\frac{3}{4}(1 - \frac{1}{2})$~~

$\boxed{E(X/Y=\tfrac{1}{2}) = 3/4}$

Q5 Let X and Y be jointly Gaussian random variables with zero mean and covariance matrix

$$C = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Define the random variables W and Z by

$$\begin{aligned} W &= X + Y \\ Z &= 2X + 3Y \end{aligned} \qquad \begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = G\begin{bmatrix} X \\ Y \end{bmatrix}$$

$\boxed{15}$ **a) Find the joint pdf of W and Z? Are these two variables independent?**

W & Z are jointly Gaussian. To determine joint pdf, we need mean & covariance

$$\begin{bmatrix} E[W] \\ E[Z] \end{bmatrix} = G\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \qquad C_{WZ} = GCG^H = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 14 \end{bmatrix} \Rightarrow \det(C_{WZ}) = 3$$

$$C_{WZ}^{-1} = \frac{1}{3}\begin{bmatrix} 14 & -5 \\ -5 & 2 \end{bmatrix}$$

$$f(w,z) = \frac{1}{\sqrt{(2\pi)^2 \det(C_{WZ})}} \exp\left(\frac{1}{6}[w \ z]\begin{bmatrix} 14 & -5 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix}\right)$$

They are not indep. bec. $E[WZ] = -5 \neq E[W]E[Z]$

$\boxed{10}$ **b) Find the mean square estimate of W given that $Z = -2$.**

Since W & Z are jointly Gaussian, the best estimate of W given Z is linear in Z & is given by

$$E[W|Z=-2] = aZ\Big|_{Z=-2} = \frac{E[ZW]}{E[Z^2]} Z\Big|_{Z=-2}$$

$$= \frac{5}{14} \times -2 = -\frac{10}{14}$$

$\boxed{19}$ **c) Bonus** Find the joint pdf of X and W.

Note that

$$\begin{bmatrix} X \\ W \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow X \text{ \& } W \text{ are jointly Gaussian}$$

They are zero mean

$$C_{XW} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad C_{XW}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det(C_{XW}) = 3$$

$$f(x,w) = \frac{1}{\sqrt{(2\pi)^2 3}} \exp\left(\frac{-1}{6}[x \ w]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix}\right)$$

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