EE 570
HW # 6
SOLUTION

TOTAL (90 points)

Q1 (10 points) (5 each)

(a)

$$x(t) = y'(t) + 2y(t) \Rightarrow h(t) = e^{-2t}U(t)$$

$$E\{|y(t)|^2\} = q_1(t) * |h(t)|^2$$

$$= 5 * e^{-4t}U(t)$$

$$= \int_0^\infty 5 \cdot e^{-4\tau}\, d\tau = \frac{5}{4}$$

(b)

for $t>0$, $q_1(t) = 5U(t)$

$$E\{y^2(t)\} = 5U(t) * e^{-4t}U(t)$$

$$= \int_0^t 5 \cdot e^{-4\tau}\, d\tau = \frac{5}{4}(1 - e^{-4t})$$

Q2 (15 points)

$$E\{s^2\} = E\left\{ \int_0^{10} x(t_1)\,dt_1 \int_0^{10} x(t_2)\,dt_2 \right\}$$

$$= E\left(\int_{-5}^{5} x(t_1)\,dt_1 \int_{-5}^{5} x(t_2)\,dt_2 \right)$$

$$= \int_{-5}^{5}\int_{-5}^{5} R_{xx}(t_1 - t_2)\,dt_1\,dt_2$$

$$E\{S^2\} = \int_{-10}^{10} (10-|\tau|)\, R_{xx}(\tau)\, d\tau$$

$$\text{Mean, } E\{S\} = \int_0^{10} E\{x(t)\}\, dt = \int_0^{10} 8\, dt = 80$$

$$\text{Variance, } E\{(S-M_S)^2\} = \int_{-10}^{10} (10-|\tau|)\left(R_{xx}(\tau) - M_x^2\right) d\tau$$

$$= \int_{-10}^{10} (10-|\tau|)\left(10e^{-2|\tau|}\right) d\tau$$

$$= 20 \int_0^{10} (10-\tau)\, e^{-2\tau}\, d\tau$$

Q3.

(15 points)

$$R_y(\tau) = E\left[\left(x(t+a) - x(t-a)\right)\left(x(t+\tau+a) - x(t+\tau-a)\right)\right]$$

$$= E\left[x(t+a)\,x(t+\tau+a)\right] - E\left[x(t+a)\,x(t+\tau-a)\right]$$

$$- E\left[x(t-a)\,x(t+\tau+a)\right] + E\left[x(t-a)\,x(t+\tau-a)\right]$$

$$= R_x(\tau) - R_x(\tau+2a) - R_x(\tau-2a) + R_x(\tau)$$

$$R_y(\tau) = 2R_x(\tau) - R_x(\tau+2a) - R_x(\tau-2a)$$

$$\Rightarrow \quad S_y(\omega) = 2S_x(\omega) - e^{-j2a\omega} S_x(\omega) - e^{j2a\omega} S_x(\omega)$$

$$= 2S_x(\omega)\left[1 - \cos 2a\omega\right]$$

$$= 4 \sin^2 a\omega\, S_x(\omega)$$

Q4.

(10 points) $R_{XY}(\tau) = B\tau e^{-W\tau} u(\tau)$

$$\Rightarrow S_{XY}(\omega) = B \cdot j \frac{d}{d\omega}\left(\frac{1}{W + j\omega}\right)$$

$$= \frac{B}{(W + j\omega)^2}$$

$$S_{YX}(\omega) = S_{XY}^*(\omega) = \frac{B}{(W - j\omega)^2}$$

Q5.

(20 points) 1. Valid
(4 each)

2. Invalid.
Reason - Not an even function as $\sin(-3\tau) \neq \sin(3\tau)$

3. Invalid
Reason - Imaginary part $j\omega^2$ is not odd

4. Valid

5. Valid

Q6.

(20 points) 1. (4 points) Input SNR $= \frac{R_Y(0)}{R_N(0)} = \frac{10 + 10^{-3}}{10^{-3}} = 10^4 + 1$

2. (4 points) $S_{YY}(\omega) = S_{XX}(\omega) + S_{NN}(\omega)$

$$S_{YY}(\omega) = \frac{20}{1+\omega^2} + 10^{-3}$$

3. (12 points) $S_{ZZ}(\omega) = |H(\omega)|^2 S_{YY}(\omega)$

$$= \frac{4}{(1+\omega^2)^3}\left(\frac{20}{1+\omega^2} + 10^{-3}\right)$$

$$\Rightarrow R_{ZZ}(0) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \frac{80}{(1+\omega^2)^4}\,d\omega + \int_{-\infty}^{\infty} \frac{4\times 10^{-3}}{(1+\omega^2)^3}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[25\pi + \frac{3\pi \times 10^{-3}}{2}\right] = 12.50075$$

$$S_{N'N'}(\omega) = \frac{4}{(1+\omega^2)^3}\cdot 10^{-3}$$

$$\Rightarrow R_{N'N'}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\times 10^{-3}}{(1+\omega^2)^3}\,d\omega = 0.75\times 10^{-3}$$

$$\text{Output SNR} = \frac{R_{ZZ}(0)}{R_{N'N'}(0)} = 16.667\times 10^3$$