Introduction to Discrete-Time Signals and Systems

$z$-Transform and Applications to Discrete-Time Systems

Lecture #40
The material to be covered in this lecture is as follows:

- Properties of the $z$-transform
  - Linearity
  - Initial and final value theorems
  - Time-delay
- $z$-transform table
- Inverse $z$-transform
- Application of $z$-transform to discrete-time systems
After finishing this lecture you should be able to:

- Find the $z$-transform for a given signal utilizing the $z$-transform tables
- Utilize the $z$-transform properties like the initial and final value theorems
- Find the inverse $z$-transform.
- Utilize $z$-transform to perform convolution for discrete-time systems.
Derivation of the \( z \)-Transform

- The \( z \)-transform is defined as follows:

\[
X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n}
\]

- The coefficient \( x(nT) \) denote the sample value and \( z^{-n} \) denotes that the sample occurs \( n \) sample periods after the \( t=0 \) reference.
- Rather than starting form the given definition for the \( z \)-transform, we may build a table for the popular signals and another table for the \( z \)-transform properties.
- Like the Fourier and Laplace transform, we have two options either to start from the definition or we may utilize the tables to find the proper transform.
- The next slide illustrates a few \( z \)-transform pairs.
- Then we will investigate some of the \( z \)-transform properties:
  - Linearity
  - Time-shifting property
  - Initial and final value theorems
# Table of $z$-transform pairs

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\delta(t - k \Delta T)$</td>
<td>$z^{-k}$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{z}{z - 1}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\frac{\Delta T z}{(z - 1)^2}$</td>
</tr>
<tr>
<td>$t^2$</td>
<td>$\frac{\Delta T^2 z (z + 1)}{(z - 1)^3}$</td>
</tr>
<tr>
<td>$e^{-at}$</td>
<td>$\frac{z}{z - e^{-a\Delta T}}$</td>
</tr>
<tr>
<td>$te^{-at}$</td>
<td>$\frac{\Delta T z e^{-a\Delta T}}{(z - e^{-a\Delta T})^2}$</td>
</tr>
<tr>
<td>$a^n u[n]$</td>
<td>$\frac{z}{z - a}$</td>
</tr>
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</table>
Linearity of the $z$-Transform

If
$$x_1[n] \leftrightarrow X_1(z) \quad \text{with region of convergence, } ROC=R_1.$$
and
$$x_2[n] \leftrightarrow X_2(z) \quad \text{with region of convergence, } ROC=R_2.$$
Then
$$a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z) \quad \text{with } ROC = R_1 \cap R_2$$

- This follows directly from the definition of the $z$-transform because the summation operator is linear.
- It is easily extended to a linear combination of an arbitrary number of signals.
- This property includes the multiplication by constant property which states that if the signal is scaled by a constant its $z$-transform will be scaled by the same constant.

$$a x_1[n] \leftrightarrow a X_1(z)$$
Time-Shifting property for the z-Transform

If
\[ x[n] \leftrightarrow X(z) \quad ROC=R \]
Then
\[ x[n-n_0] \leftrightarrow z^{-n_0}X(z) \quad ROC=R \]

Proof
\[ Z \{x[n-1]\} = \sum_{n=-\infty}^{\infty} x[n-1]z^{-n} \]
\[ = z^{-1} \sum_{n=-\infty}^{\infty} x[n-1]z^{-(n-1)} \]
\[ = z^{-1} \sum_{m=-\infty}^{\infty} x[m]z^{-m} = z^{-1}Z\{x[n]\} \]

This property will be very important for producing the \textbf{z-transform transfer function of a difference equation} which uses the property:
\[ x[n-1] \leftrightarrow z^{-1}X(z) \]
Example 40.1: Properties of the $z$-transform

Find the $z$-transform for the input signal

$$x[n] = 7(1/3)^{n-2}u[n - 2] - 6(1/2)^{n-1}u[n - 1]$$

Solution:

We know that

$$a^n u[n] \leftrightarrow \frac{z}{z - a}$$

So

$$X(z) = 7z^{-2} \frac{z}{z - 1/3} - 6z^{-1} \frac{z}{z - 1/2}$$

$$= 7 \frac{1}{z^2 - 1/3z} - 6 \frac{1}{z - 1/2}$$
Initial and Final Value Theorems
If $x[n]$ has a $z$-transform $X(z)$ and if $\lim_{z \to \infty} X(z)$ exists, then

$$\lim_{n \to 0} x[n] = x[0] = \lim_{z \to \infty} X(z)$$

This theorem can be easily proven by the definition of the $z$-transform

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

As we take the limit all terms will be zero except the first term

The final value theorem which is given by

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} \left( (1 - z^{-1})X(z) \right)$$
Example 40.2: Application of the initial and final value theorems

Find the initial and final values for the following signal expressed in its z-transform

\[ F(z) = \frac{0.792z^2}{(z - 1)(z^2 - 0.416z + 0.208)} \]

Solution:

Initial-value

\[ F(z \to \infty) = \frac{0.792z^2}{z^3} = 0 \]

Final -value

\[ f(n \to \infty) = \frac{0.792}{(1-0.416+0.208)} = 1 \]

These answers can be justified by looking at the expansion of the given expression

\[ F(z) = 0.792z^{-1} + 1.12z^{-2} + 1.091z^{-3} + 1.01z^{-4} + 0.983z^{-5} + 0.989z^{-6} + 0.99z^{-7} \ldots \]

- The coefficient for \( z^0 \) is zero which is the initial value.
- The coefficient converges to one as the negative power of \( z \) increases which corresponds to the final value.
• The inverse operation for the z-transform may be accomplished by:
  o Long division
  o Partial fraction expansion
• The z-transform of a sample sequence can be written as
  \[ X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots. \]
• If we can write \( X(z) \) into this form, the sample values can be determined by inspection.
• When \( X(z) \) is represented in a ratio of polynomials in \( z \), this can be easily achieved by long division.
• Before carrying out the division, it is convenient to arrange both the numerator and the denominator in ascending powers of \( z^{-1} \).
Inverse $z$-transform using Partial Fraction Expansion

- Alternatively, we may avoid the long division by partial fraction expansion. The idea is similar to the method used for inverse Laplace transform.
- The objective is to manipulate $X(z)$ into a form that can be inverse $z$-transformed by using $z$-transform tables.
- Because of the forms of transforms,
  - it is usually best to perform partial fraction expansion of $H(z)/z$.
  - As an alternative $z^{-1}$ can be treated as the variable in the partial fraction expansion.
- Important: before doing partial-fraction expansion, make sure the $z$-transform is in proper rational function of $z^{-1}$!
Example 40.3: Inverse $z$-Transform Using Partial Fraction Expansion

Find the inverse $z$-transform using both partial fraction expansion and long division

$$X(z) = \frac{z^2}{(z-1)(z-0.2)}$$

**Solution:**

If we treat $z^{-1}$ as the variable in the partial fraction expansion, we can write

$$X(z) = \frac{1}{(1-z^{-1})(1-0.2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$$

Utilizing Heaviside’s Expansion Method:

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$$

(1) 

$$(1-z^{-1})X(z) = \frac{1}{1-0.2z^{-1}} = A + \frac{B(1-z^{-1})}{1-0.2z^{-1}} \Rightarrow$$

$$\frac{1}{1-0.2} = A + \frac{B \cdot 0}{1-0.2} \Rightarrow A = \frac{1}{0.8} = 1.25$$

(2) 

$$(1-0.2z^{-1})X(z) = \frac{A(1-0.2z^{-1})}{1-z^{-1}} + \frac{B \cdot (1-0.2z^{-1})}{1-0.2z^{-1}} \Rightarrow$$

$$\frac{1}{1-z^{-1}} = \frac{A(1-0.2z^{-1})}{1-z^{-1}} + B$$
Continue ...Example 40.3

\[ 1-0.2z^{-1}=0 \ (z=0.2) \]
\[ \Rightarrow \quad B = 1/(1 - 5) = -1/4 = -0.25 \]

\[ X(z) = \frac{1.25}{1-z^{-1}} + \frac{-0.25}{1-0.2z^{-1}} \Rightarrow x(nT) = 1.25 - 0.25(0.2)^n \]

From which we may find that \( x(0)=1, x(T)=1.2, x(2T)=1.24, x(3T)=1.248 \)

We may get to the same answer using long division. \( X(z) \) is written as

\[ X(z) = \frac{z^2}{z^2 - 1.2z + 0.2} \]

which is, after multiplying numerator and denominator by \( z^{-2} \)

\[ X(z) = \frac{1}{1-1.2z^{-1}+0.2z^{-2}} \]

Now, it is left for you to show that the long division will result in the same answer given by

\[ X(z) = 1 + 1.2z^{-1} + 1.24z^{-2} + 1.248z^{-3} + \ldots \]
Discrete-Time Systems

- For continuous-time systems, differential equation may solved using Laplace transform.
- Similarly discrete-time systems result in *Difference Equations* which may be solved using *z*-transform.
- Recall that discrete-time systems process a discrete-time input signal to produce a discrete-time output signal.
- The general symbolic notation for Discrete-Time System:

\[ y(nT) = H[x(nT)] \]

- Similar to continuous-time systems we may define some properties for the discrete-time systems. For example,

1. **Shift-invariant system:** a system is shift invariant if \( H[x(nT - n_0T)] = y(nT - n_0T) \) true for any \( n_0 \). (Similar to the concept of time-invariant systems for continuous-time)
2. **Causal and noncausal systems:** physical Description: A system is causal or nonanticipatory if the system’s response to an input does not depend on future values of the input.
3. **Linear System**
   - Linear System \( \Leftrightarrow H[\alpha_1 x_1(nT) + \alpha_2 x_2(nT)] = \alpha_1 H[x_1(nT)] + \alpha_2 H[x_2(nT)] \)
Application of the $z$-Transform to Linear Discrete-Time Signals

Linear Systems: can be modeled as

$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT) \quad \text{or} \quad y(nT) = \sum_{k=-\infty}^{\infty} h(kT)x(nT - kT)$$

$\text{Convolution}$

$h(kT)$: response of the shift-invariant linear system at $t=kT$ to an impulse input applied at $t=0$.

Causal systems: $h(kT) = 0 \quad \forall k < 0$

If the system is linear and causal in addition to the fact that $x(kT) \equiv 0 \quad (k < 0)$ then

$$\Rightarrow y(nT) = \sum_{k=0}^{\infty} x(kT)h(nT - kT)$$

$$\quad x(kT)=0(k<0) \quad = \quad \sum_{k=0}^{\infty} x(kT)h(nT - kT)$$

$$(nT - kT \geq 0) \quad nT \geq kT \quad \rightarrow \quad = \sum_{k=0}^{n} x(kT)h(nT - kT) = \sum_{k=0}^{n} h(kT)x(nT - kT)$$
Transfer Function in the z-Domain

- The $z$-transform is **linear**
- There is a simple relationship for a signal **time-shift**
  \[ x[n-1] \leftrightarrow z^{-1}X(z) \]
- This is fundamental for deriving the transfer function of a difference equation which is expressed in terms of the input-output signal delays
- The **transfer function** of a discrete time LTI system is the $z$-transform of the system’s impulse response
- The transfer function is a rational polynomial in the complex number $z$.
- Convolution is expressed as multiplication
  \[ x[n] * h[n] \leftrightarrow X(z)H(z) \]
and this can be solved for particular signals and systems
Example 40.4: Discrete-Time Convolution

Calculate the output of a first order difference equation of a input signal

\[ x[n] = 0.5nu[n] \]

\[ 0.5^*u[n] \leftrightarrow X(z) = \frac{z}{z-0.5} \]

System transfer function (z-transform of the impulse response)

\[ y[n] - 0.8y[n-1] = x[n] \]

Taking the z-transform of the difference equation

\[ Y(z) - 0.8z^{-1}Y(z) = X(z) \]

\[ Y(z)(1 - 0.8z^{-1}) = X(z) \]

\[ H(z) = \frac{1}{1 - 0.8z^{-1}} = \frac{z}{z-0.8} \]

The (z-transform of the) output is therefore the product:

\[ Y(z) = \frac{z^2}{(z-0.5)(z-0.8)} \]

\[ = \frac{1}{0.3} \left( \frac{0.8z}{z-0.5} - \frac{0.5z}{z-0.8} \right) \quad \text{ROC} \quad |z|>0.8 \]

\[ y[n] = (0.8*0.5^*u[n]-0.5*0.8^*u[n])/0.3 \]
Self Test
Question 1:
\[ Z(x(nT)) = \frac{1}{1 + 0.5z^{-1}} \]
If \( Z(x(nT - 2T)) \), what’s \( Z(x(nT - 2T)) \)?
Answer: \( \frac{z^{-2}}{1 + 0.5z^{-1}} \)

Question 2:
Find the z-transform for \( Y(z) = \frac{1}{z^2 - 1.2z + 0.2} \)
Answer: \( y(nT) = 5\delta(n) + 1.25 - 6.25(0.2)^n \) (Try to find a relation with example 40.3)

Question 3:
\[ x(nT) = \left(\frac{1}{4}\right)^n u(n-3) = \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^{n-3} u(n-3) \]
\[ h(nT) = \left(\frac{1}{3}\right)^n u(n-5) = \left(\frac{1}{3}\right)^5 \left(\frac{1}{3}\right)^{n-5} u(n-5) \]
Find \( x(nT)*h(nT) \)
Answer: \( x(nT)*h(nT) = \left(\frac{1}{4}\right)^3 \left(\frac{1}{3}\right)^5 \left[4\left(\frac{1}{3}\right)^{n-8} - 3\left(\frac{1}{4}\right)^{n-8}\right] u(n-8) \)
Question 4:
Calculate the step response to the system describe by the following difference equation
\[ 6y[n] - 5y[n - 1] + y[n - 2] = x[n] \]

Answer
\[ u[n] \leftrightarrow X(z) = \frac{1}{1 - z^{-1}} \]
\[ H(z) = \frac{1}{6 - 5z^{-1} + z^{-2}} = \frac{1}{(2 - z^{-1})(3 - z^{-1})} \]
\[ Y(z) = \frac{1}{(3 - z^{-1})(2 - z^{-1})(1 - z^{-1})} \]
\[ = 0.5 \frac{1}{(3 - z^{-1})} - \frac{1}{(2 - z^{-1})} + 0.5 \frac{1}{(1 - z^{-1})} \]
\[ = 0.167 \frac{1}{(1 - 1/3 z^{-1})} - 0.5 \frac{1}{(1 - 1/2 z^{-1})} + 0.5 \frac{1}{(1 - z^{-1})} \]
\[ y[n] = (0.167(1/3)^n - 0.5(1/2)^n + 0.5)u[n] \]