Introduction to Discrete-Time Signals and Systems

The $z$-Transform

Lecture #39
The material to be covered in this lecture is as follows:

- Introduction to the $z$-transform
- Definition of the $z$-transform
- Derivation of the $z$-transform
- Region of convergence for the transform
- Examples.
After finishing this lecture you should be able to:

- Find the $z$-transform for a given signal utilizing the $z$-transform definition
- Calculate the region of convergence for the transform
Derivation of the $z$-Transform

- The $z$-transform is the basic tool for the analysis and synthesis of discrete-time systems.
- The $z$-transform is defined as follows:

$$X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n}$$

- The coefficient $x(nT)$ denote the sample value and $z^{-n}$ denotes that the sample occurs $n$ sample periods after the $t=0$ reference.
- Rather than starting form the given definition for the $z$-transform, we may start from the continuous-time function and derive the $z$-transform. This is done in the next slide.
Derivation of the $z$-transform

The sampled signal may be written as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t-nT)$$

Since $\delta(t-nT) = 0$ for all $t$ except at $t=nT$, $x(t)$ can be replaced by $x(nT)$. Assuming $x(t)=0$ for $t<0$. Then,

$$x_s(t) = \sum_{n=0}^{\infty} x(nt) \delta(t-nT)$$

Taking Laplace transform yields

$$X_s(s) = \int_0^{\infty} \sum_{n=0}^{\infty} x(nt) \delta(t-nT) e^{-st} dt$$

Rearranging

$$X_s(s) = \sum_{n=0}^{\infty} x(nT) \int_0^{\infty} \delta(t-nT) e^{-st} dt$$

By sifting property of the delta function

$$X_s(s) = \sum_{n=0}^{\infty} x(nT) e^{-snT}$$
Continue Derivation…

Defining the complex variable \( z \) as the Laplace time-shift operator

\[
    z = e^{sT}
\]

Hence,

\[
    X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n}
\]

We could have started from here but it is good to relate to the s-domain.

In the s-domain the left-half plane corresponds to \( \sigma < 0 \) is mapped to \( |z| < 1 \) in the z-plane which is the region inside the unit circle.
Region of Convergence (ROC)

$|z|$ is converged for $\sigma < 0$ (left-half of $s$-plane). This corresponds to $|z|<1$. This is the region inside the unit circle.

$|z|$ is NOT converged for $\sigma > 0$ (right-half of $s$-plane). This corresponds to $|z|>1$ which is the region outside the unit circle.

$$z = e^{sT}$$

$$s = \sigma + j \omega$$

$$z = e^{\sigma T} e^{j \omega T}$$

$$|z| = e^{\sigma T}$$

The mapping of the Laplace variable $s$ into the $z$-plane through $z=e^{sT}$ is illustrated in the figure below:
The Z-Transform in Summary

\[ X(z) = \sum_{n=0}^{\infty} x(nT)z^{-n} \quad \text{where} \quad z = e^{sT} \]

- The coefficient \( x(nT) \) denotes the sampled value
- \( z^{-n} \) denotes that the sample occurs \( n \) sample periods after the \( t=0 \) reference.
- \( e^{sT} \) is simply the \( T \)-second time shift
- The parameter \( z \) is simply shorthand notation for the Laplace time shift operator
- For instance, \( 30z^{-40} \) denotes a sample, having value 30, which occurs 40 sample periods after the \( t=0 \) reference
- The definition of \( z \)-transform can also be written as: (other text books)
  \[ X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \quad \text{for} \quad n \geq 0 \]
  where the square bracket is used to indicate discrete times.
- It worth to mention that Matlab has special tools for Z-transform.
Example 39.1
Determine the $z$-transform for the following signal

$$x[n] = \begin{cases} 
1, & n = -1 \\
2, & n = 0 \\
-1, & n = 1 \\
1, & n = 2 \\
0, & \text{otherwise}
\end{cases}$$

• Solution:
We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

hence

$$X(z) = \sum_{n=1}^{2} x[n]z^{-n} = x[-1]z^{-(-1)} + x[0]z^{-0} + x[1]z^{-1} + x[2]z^{-2}$$

$$X(z) = z + 2 + z^{-1} + z^{-2}$$
Example 39.2: Sampled Step Function (*Important Functions*)

Consider a unit step sample sequence defined by

\[ x[n]=1, \quad n \geq 0 \]

Find the z-transform.

**Solution:**

\[ U(z) = X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \quad |z| > 1 \]

The sum converges absolutely to \(1/(1-z^{-1})\) outside the unit circle \(|z|>1\)

\[ X(z) = \sum_{n=0}^{\infty} z^{-n} \]
Sampled Dirac Delta Function (*an other important function*)

The Dirac Delta Function is defined to be
\[
\delta(n) = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 
\end{cases}
\]

For a delayed version of delta is defined as
\[
\delta(n-k) = \begin{cases} 
1 & n = k \\
0 & n \neq k 
\end{cases}
\]

Applying the definition of the $z$-transform

\[
X(z) = \sum_{k=0}^{\infty} \delta(t)z^{-s \Delta T} = \delta(0) = 1
\]

\[
X(z) = 1
\]
The Unit Exponential Sequence

The unit exponential sequence is defined to be

\[ x(k) = \begin{cases} e^{-\alpha k} & k, \alpha > 0 \\ 0 & k < 0 \end{cases} \]

Apply z-transform definition

\[ X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} \]

we get

\[ X(z) = \sum_{k=0}^{\infty} e^{-\alpha k}z^{-k} = \sum_{k=0}^{\infty} (e^{-\alpha}z^{-1})^k \]

\[ X(k) = \frac{1}{1-e^{-\alpha}z^{-1}} = \frac{z}{z-e^{-\alpha}} \]

where \(|z| > e^{-\alpha}\)

if \(k = e^{-\alpha}\) then

\[ X(z) = \frac{1}{1-kz^{-1}} = \frac{z}{z-k} \]
Example 39.3
Determine the $z$-transform of the signal
\[ x[n] = 0.5^n u[n] \]
Depict the ROC and the locations of poles and zeros of $X(z)$ in the $z$-plane

Solution:
Substituting is the definition of the $z$-transform
\[ X(z) = \sum_{n=0}^{\infty} \left( \frac{0.5}{z} \right)^n \]
This is a geometric series of infinite length in the ratio $0.5/z$; the sum converges, provided that $|0.5/z| < 1$ or $|z| > 0.5$. Hence the $z$-transform is
\[ X(z) = \sum_{n=0}^{\infty} \left( \frac{0.5}{z} \right)^n = \frac{1}{1-0.5z^{-1}}, \quad |z| > 0.5 \]
\[ = \frac{z}{z-0.5}, \quad |z| > 0.5 \]

**Pole at $z=0$, zero at $z=0.5$, ROC is the light blue region**
Self Test:
1) Determine the $z$-transform for the following signal

$$x[n] = \left\{ \left( \frac{1}{2} \right)^n u[n] \right\}$$

Solution:
• Utilizing the definition of the $z$-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

• Hence

$$X(z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2} \right)^n u[n] \quad z^{-n} = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \quad z^{-n}$$

$$X(z) = \left( \frac{1}{2} \right)^0 z^{-0} + \left( \frac{1}{2} \right)^1 z^{-1} + \left( \frac{1}{2} \right)^2 z^{-2} + ...$$

$$X(z) = 1 + \left( \frac{1}{2z} \right) + \left( \frac{1}{4z^2} \right) + ...$$
2) Find the $z$-transform of the following signal:

$$X(nT) = 0.5^n \cos\left(\frac{n\pi}{2}\right)$$

Hint: Click to show hint: $\cos\left(\frac{n\pi}{2}\right) = 0$ for $n$ odd & $\pm 1$ for even values of $n$.

Answer: Click to show answer: $X(z) = \frac{1}{1 + a^2 z^{-2}}$  $|z| > |a|$

3) Determine the $z$-transform of the signal

$$x[n] = -u[-n-1] + 0.5n u[n]$$

Depict the ROC and the locations of poles and zeros of $X(z)$ in the $z$-plane

Solution: Click to show answer:

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n - \sum_{n=-\infty}^{-1} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{0.5}{z}\right)^n + 1 - \sum_{k=0}^{\infty} z^k$$

the sum converges, provided that $|z| > 0.5$ and $|z| < 1$.  

\[ X(z) = \frac{1}{1-0.5z^{-1}} + 1 - \frac{1}{1-z}, \quad 0.5 < |z| < 1 \]

\[ = \frac{z (2z - 1.5)}{(z - 0.5)(z - 1)}, \quad 0.5 < |z| < 1 \]

*Poles at \( z=0.5, 1 \), zeros at \( z=0, 0.75 \). ROC is the region in between*