2009 IEEE International Conference on Robotics and Automation
Kobe International Conference Center
Kobe, Japan, May 12-17, 2009

A Harmonic Potential Field Approach with a Probabilistic Space Descriptor for Planning in Non-divisible Environments.

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Abstract: This paper extends the capabilities of the harmonic potential field approach to planning to cover the situation where the workspace of a robot cannot be segmented into geometrical subregions each having an attribute of its own. Instead the suggested planner accepts a holistic, task-centered, probabilistic descriptor of the workspace as an input. This descriptor is processed along with a goal point to yield the navigation policy used to direct motion. The extension is based on the physical analogy with an electric current flowing in a nonhomogeneous conducting medium. Proofs of the ability of the modified approach to avoid zero-probability (definite threat) regions and converge to the goal are provided. The capabilities of the suggested planner are demonstrated using simulation.

I. Introduction:

Designing an autonomous agent that can, with a reasonable chance of success, engage a task in a realistic environment is an involved multi-disciplinary endeavor [1]. While special attention has to be paid to the propulsion, data acquisition and communication systems used by the agent, the biggest challenge seems to be designing a proper planning module that converts these systems into one goal-oriented unit. There is a long list of conditions a planner must satisfy in order to generate a sequence of action instructions which the actuators of motion may execute to successfully complete a task. However, the conditions on handling and representing mission data seem to be the most stringent [2,33]. To begin with, the format of the signal encoding the information acquired about the agent's environment has to be carefully selected. Two core conditions on a representation are: compatibility of the representation with the manner in which the data is being processed and action is being generated. This condition minimizes the chance of inter-module conflict, hence reduces the probability of unexpected behavior. It also reduces the delay in responding to the contents of the environment. The other condition has to do with the updatability of the representation. No matter how accurate the representation an agent has prior to executing a task, it will have to change during the execution phase depending on the feedback received from the agent's sensors. The validity of the existing portion of the representation must not be conditioned on the future data that could be received. In other words, the presentation must be noncommittal in order for the size of the effort needed to incorporate a newly acquired piece of information to be commensurate with the size of the newly available information. This allows for the incremental construction of a representation. Besides these two requirements, it is desirable that the growth in the size of a representation be linear (or a low order polynomial) in the size of information (assuming it is possible to construct an information measure). It is also useful for a representation to be able to incorporate the ambiguity caused by the aging of information during the period where no sensory data is available to update the agent's belief of its environment. The ability to incorporate human-centered, intuitive, expert knowledge in a representation is also important.

Most planners assume divisible environments that may be partitioned into subsets of homogeneous attributes. The most common scheme is to have a binary partition of admissible sets and forbidden ones. These regions are usually described using geometric structures like circles [3], occupancy maps, Voronoi partitions [4], grids and graphs [5], and trees [6]. This crisp representation is then fed to the planner along with the goal (or task) and constraints on behavior in order to generate the action instructions needed by the agent being utilized to perform the task. There are situations where the environment in which an agent is operating is not divisible. For example, a plane flying through turbulent atmosphere [7] experiences a degree of turbulence where clear space is diffused into turbulent space with no sharp boundaries separating the two. Also in the case of mobile robots operating in rough terrains [8], it is not wise to construct a binary description of the environment using admissible and forbidden regions. Instead, the description should be based on the degree of difficulty of negotiating the terrain. Many other examples similar to those can be constructed like passing through hostile space where the environment is better described by the degree of possible harm.

An alternative to the geometric approach is to use a soft representation that consists of a field reflecting, at each point in the environment, the ability of achieving the task. Describing an environment by a task-centered, probability field can address almost all the above requirements. For example, aging of information may be incorporated by using a simple blurring operator. Exact knowledge of the location of a one-dimensional point object may be represented as a probability distribution function (PDF) with an impulse function (figure-1). Convolving this exact spatial knowledge with the proper blurring operator produces another representation with the ambiguity factored-in. Repeatedly applying the blurring operator leads to a uniform PDF representing the maximum state of ambiguity (i.e. the object could be anywhere). Probabilistic representations are ideal for encoding the information in non-divisible environments. For example the speed of wind in a storm (figure-2) may be easily converted into a PDF representing the suitability of space for navigation.

As for the amount of storage needed for a faithful reproduction of the representation, it is well-known from Shannon sampling theorem [9] that there is no need to store the value of the PDF at each point in space. Depending on the bandwidth (BW) of the PDF (a BW may in many cases be used as a measure of the richness of a signal in information) only sparse information about the pulse-like interpolating kernel need to be stored to exactly reproduce the PDF. Moreover, a change of information in one region of the PDF will not affect the information in the other regions. This allows for the representation to be, with reasonable effort, dynamically updated.

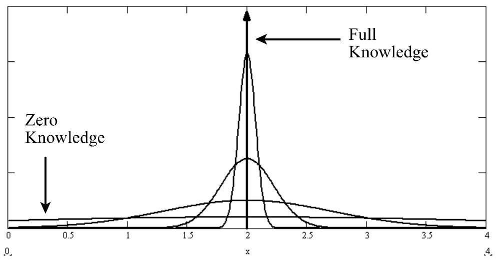


Figure-1: Probabilistic descriptor easily admits aging of information.

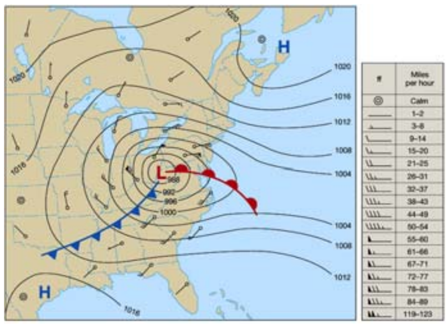


Figure-2: Speed of wind may be used to construct a spatial probabilistic descriptor.

Soft probabilistic representations were adopted by many researchers in robotics. For example in [10,11] probabilistic representations are used for path planning in the presence of sensory data ambiguity. This representation may be fed to a reinforcement learning-based, or an optimal control-based stage to generate the navigation policy.It ought to be noticed that a representation generated in this manner reflects the structure of the underlaying physical environment (figure-3). If the probabilistic field is to accommodate all the above mentioned properties, the chances are that the resulting PDF will not reflect this structure. This in turn will adversely affect the performance of the navigation policy generator.

Fuzzy logic techniques [12] are used to derive navigation actions for robots. They use soft probabilistic representations in the process. While these techniques have proven their practicality, they are not provably correct (heuristics). They heavily rely on an external user's understanding of the nature

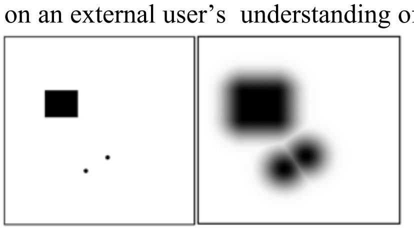


Figure-3: Ambiguity function reflects the crisp geometry of the environment.

of the environment being tackled in order to generate the fuzzy rule-book needed for the navigator to function. This may seriously jeprodize the autonomy of the robot. Qualitative methods for dealing with uncertainty may also be found in [13].

This work suggests a method for generating the navigation policy for a robot. The method accepts as an input the goal point and a task-centered, probabilistic description of the environment. The method is efficient, provably-correct and assumes no structure what so ever on the PDF used to represent the environment. The method is derived by extending the capabilities of the harmonic potential field (HPF) approach to motion planing [14,15,16] so that it would be possible to feed the environmental data to the planner in the form of a PDF instead of a geometric form. Previous attempts to constructing a probabilistic HPF planner focused on using the HPF as a probability measure describing the danger of collision with obstacles in the environment. This approach strips the HPF from its well justified role as a navigation policy generator and reduces it to a merely descriptive tool that is an input to an action generation stage of some kind [17,18].

This paper is organized as follows: in section II, the suggested planner is presented. The ability of the planner to avoid zero-probability regions and converge to the target are proven in section III. Simulation results are in section IV and conclusions are placed in section V.

II. The Suggested extension :

Harmonic potential fields (HPFs) have proven themselves to be effective tools for inducing in an agent an intelligent, emergent, embodied, context-sensitive and goal-oriented behavior (i.e. a planning action). A planning action generated by an HPF-based planner can operate in an informationally-open and organizationally-closed mode [19] enabling an agent to make decisions on-the-fly using on-line sensory data without relying on the help of an external agent. HPF-based planners can also operate in an informationally-closed, organizationally-open mode which makes it possible to utilize existing data about the environment in generating the planning action as well as elicit the help of external agents . A hybrid of the two modes may also be constructed. Such features make it possible to adapt HPFs for planning in a variety of situations. For example in [20] vector-harmonic potential fields were used for planning with robots having second order dynamics. In [21] the approach was configured to work in a pursuit-evasion planning mode, and in [22] the HPF approach was modified to incorporate joint constraints on regional avoidance and direction. The decentralized, multi-agent, planning case was tackled using the HPF approach in [23]. The HPF approach was also found to facilitate the integration of planners as subsystems in networked controllers containing sensory, communication and control modules with a good chance of yielding a successful behavior in a realistic, physical setting [24]. A basic setting of the HPF approach is shown in (1) below:

$$\nabla^2 V(X) \equiv 0 \qquad X \in \Omega \tag{1}$$

subject to: $V(X_S) = 1$, $V(X_T) = 0$, and $\dfrac{\partial V}{\partial \mathbf{n}} = 0$ at $X = \Gamma$,

A provably-correct path may be generated using the gradient dynamical system:

$$\dot{x} = -\nabla V(x). \tag{2}$$

where Ω is the workspace, Γ is its boundary, $\mathbf{n}$ is a unit vector normal to Γ, X_s is the start point, and X_T is the target point. The above equations model the behavior of an electric current flowing in a homogeneous conductor with a conductivity $\sigma(x)$= constant [25]. The conductor is populated by perfect insulators (σ=0) occupying the forbidden regions surrounded by Γ (figure-4).

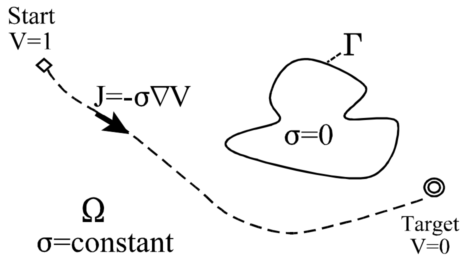


Figure-4: Physical metaphor for the planner

The flow is described by the electric current density(J):

$$J(x) = -\sigma(x)\nabla V(x). \tag{3}$$

The homogeneous Neumann condition means that the current cannot penetrate the perfect conductor and has to move tangent to it. The Laplce equation is simply a product of applying the continuity condition to the electric flow:

$$\nabla \cdot (-\sigma(x)\nabla V(x)) = 0. \tag{4}$$

If σ is constant, equation (4) reduces to the well-known Laplace operator in (1). The conductivity σ represents how favorable a point in the robot's space is to conducting motion. A σ=0 means that the corresponding space does not support motion at all. On the other hand, a high value of σ means that the corresponding space highly favors motion.

It is possible to establish a tight analogy between the above situation and the situation where the agent's environment is to be represented in a task-centered manner using a PDF (P(x)) describing at each point in the agent's space the agent's ability to perform the assigned task. It does not matter what the causes are (e.g. sensor problems, rough terrains, man-made hazzards etc.), if the agent is expected to encounter difficulties operating, a low value for P(x) is assigned to that region. Generating a path in this situation may be done using the planner:

$$\nabla\cdot(P(x)\nabla V(X)) \equiv 0 \quad X\in\Omega \tag{5}$$

subject to: $V(X_S) = 1, \; V(X_T) = 0$

A provably-correct path may be generated using the gradient dynamical system:

$$\dot{x} = -\nabla V(x). \tag{6}$$

It is shown later in the paper that the generated path, which is guaranteed to connect the start point to the target, will strictly avoid regions assigned P(x)=0.

The modified differential operator in (5) is strongly related to the ordinary Laplace operator and possesses a useful physical interpretation. First, the operator should be expanded as:

$$\nabla\cdot(P(x)\nabla V(X)) = P(x)\nabla^2 V(X) + \nabla P(x)^t \nabla V(X) = 0, \tag{7}$$

which leads to:

$$\nabla^2 V(X) = -(1/P(x))\,(-\nabla P(x)^t(-\nabla V(X))).$$

Notice that -∇V(x) is the direction at which motion is to be driven and -∇P(x) is a vector pointing in the direction of increasing risk. Also keep in mind that the Laplacian of a potential is the divergence of the gradient field which is physically defined as: the outflow of flux when the volume shrinks to zero. If the laplacian is negative, it means that there is a sink in the closed region, i.e. motion is inhabited by absorbing the gradient flux used to direct motion. If the laplacian is positive, it means that there is a source in the closed region, i.e. motion is stimulated by aiding the flow of the gradient flux. When the laplacian is zero (Lapace equation), the region is neutral towards the gradient flux. As can be seen the modified operator can be viewed as an intelligent version of the HPF which is sensitive to the future ability of the agent to carry-out its task (figure-5).

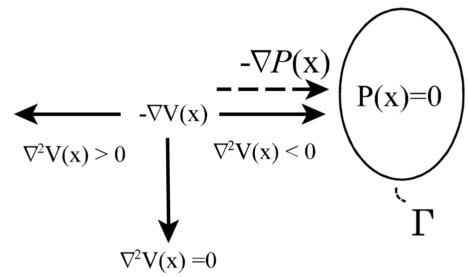


Figure-5 behavior of the modified operator.

III. Performance analysis:

In this section propositions along with their proofs are provided to explore the behavior of the suggested planning method.

Proposition-1: The path generated by the PDE-ODE system in (5) will avoid regions that have zero probability of achieving the task (O={x: P(x) = 0}).

Proof: Assume a point x that is arbitrarily close to O (figure-6). Since P(x) is differentiable its value may be assumed equal to zero. Using the identity:

$$\nabla\cdot(P(x)\nabla V(x)) = \nabla P(x)^t \nabla V(x) + P(x)\nabla^2 V(x) = 0 \tag{8}$$

when x is close to O, (8) reduces to:

$$-(-\nabla P(x)^t)(-\nabla V(x)) = 0.$$

Note that -∇P(x) points in the direction of increasing risk that leads to the region O, while -∇V(x) is the direction along which motion is to be steered. In other words, in the vicinity of O, the planner will project motion tangent to the boundary of the zero probability region; hence, O will be avoided.

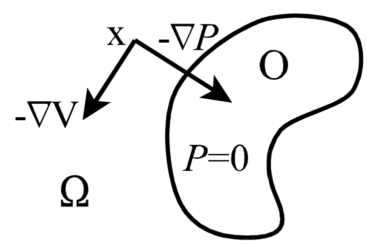


Figure-6: Zero-P regions will be avoided

Proposition-2: A potential field generated using the boundary value problem in (5) cannot have any minima local or global in its workspace Ω ($\Omega = RN-(O \cup x_T \cup x_S)$).

Proof: Note that from proposition-1 x will always stay in the workspace ($x \in \Omega$) in which P(x) is greater than zero. The fact that no minima or maxima can occur in Ω may be inferred directly from the differential operator

$$\nabla \cdot (P(x)\nabla V(x)) = 0 \qquad x \in \Omega \qquad (9)$$

At a local maxima, ∇V(x) will be negative in the whole local neighborhood surrounding x. Since P(x) in that region is finite and positive, the governing relation in (9) will be violated. Same thing will happen at a minima where ∇V(x) will be positive in the whole local neighborhood surrounding x. Therefore, no local or global minima or maxima can occur in Ω.

Proposition-3: For an Ω with a finite size, $V(x) \in \Omega$ is a Liapunov function candidate (LFC).

Proof: An LFC defined on a finite space must satisfy the followings:
1- It must be differentiable or at least continuous,
2- it must be positive in Ω ($V(x) > 0,\ x \in \Omega$),
3- its value must be zero at the target point ($V(x_T) = 0$).

Since V(x) is forced to satisfy the differential condition in (9) it is analytic. Therefore, it satisfies the first condition. Since the global maximum happens at $x=x_s$ ($V(x_s)=1$) and a global minimum at $x=x_T$ ($V(x_T)=0$) , the second and third conditions are satisfied.

Proposition-4: If V(x) is constant at a subset of Ω, it is constant for all Ω.

Proof: Assume that V(x) = C (C is a constant) in ω where $\omega \subset \Omega$ (figure-7). Consider an infinitesimally expanded region $\omega`$ that surrounds $\omega$. Let $x_o$ be a point that lies on the boundary of $\omega$ ($\partial\omega$) and $x_o^+$ a point on $\partial\omega`$.

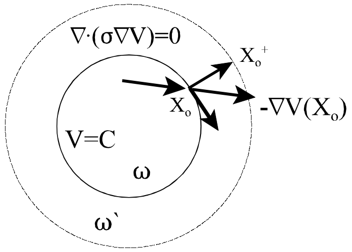


Figure-7: subregions of degenerate fields cannot occur

The potential at x_o^+ may be written as:

$$V(x_o^+) = V(x_o) + dr \cdot (-\nabla V(x_o)^t n) \qquad (10)$$

where dr is a differential element and n is a unit vector normal to $\partial\omega$. Since V is constant inside ω the gradient field degenerates to zero. Since the continuity relation (9) is enforced in both ω and $\omega`$, equation (10) reduces to:

$$V(x_o^+) = V(x_o) = C\,. \qquad (11)$$

By repeatedly applying the above procedure, the subregion ω may be expanded to include all Ω. In other words, if V(x) is constant on a subregion of Ω it will be constant for all Ω.

Definition-1: Let V(X) be a smooth (at least twice differentiable) scalar function ($V(X): R^N \to R$). A point Xo is called a critical point of V if the gradient vanishes at that point (∇V(Xo)=0); otherwise, Xo is regular. A critical point is Morse, if its Hessian matrix (H(Xo)) is nonsingular. V(X) is Morse if all of its critical points are Morse [27].

Proposition-5: If V(X) is a function defined in an N-dimensional space (R^N) on an open set Ω and satisfies (9), then the Hessian matrix at every critical point of V is nonsingular, i.e. V is Morse.

*Proof***:** There are two properties of V that are used in the proof:

1- V(X) defined on an open set Ω contains no maxima or minima, local or global in Ω. An extrema of V(X) can only occur at the boundary of Ω,
2- if V(X) is constant in any open subset of Ω, then it is constant for all Ω.

Let Xo be a critical point of V(X) inside Ω. Since no maxima or minima of V exist inside Ω, Xo has to be a saddle point. Let V(X) be represented in the neighborhood of Xo using a second order Taylor series expansion:

$$V(X) = V(Xo) + \nabla V(Xo)^T (X - Xo) + \frac{1}{2}(X - Xo)^T H(Xo)(X - Xo) \qquad |X\text{-}X0| << 1. \qquad (12)$$

Since Xo is a critical point of V, we have:

$$V^{`} = V(X) - V(Xo) = \frac{1}{2}(X - Xo)^T H(Xo)(X - Xo) \quad |X\text{-}X0| << 1. \quad (13)$$

Notice that adding or subtracting a constant to V yields another potential field that satisfies the relation in (9). Using eigenvalue decomposition [29]:

$$V^{`} = \frac{1}{2}(X - Xo)^T U^T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_N \end{bmatrix} U(X - Xo) = \frac{1}{2}\xi^T \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & . & 0 \\ . & . & . & . \\ 0 & 0 & . & \lambda_N \end{bmatrix} \xi = \frac{1}{2}\sum_{i=1}^{N} \lambda_i \xi_i^2 \quad (14)$$

where U is an orthonormal matrix of eigenvectors, λ's are the eigenvalues of H(Xo), and $\xi = [\xi_1\ \xi_2\ ..\xi_N]^T = U(X\text{-}Xo)$. Since V` cannot be zero on any open subset Ω, otherwise, it will be zero for all Ω, which is not the case. This can only be true if and only if all the λ_i's are nonzero. In other words, the Hessian of V at a critical point Xo is nonsingular. This makes V a Morse function.

Proposition-6: Let V(x) be the potential field generated using the BVP in (5). The trajectory of the dynamical system:

$$\dot{\mathbf{x}} = -\nabla \mathbf{V}(\mathbf{x}) \qquad (15)$$

will globally, asymptotically converge to:

$$\lim_{t \to \infty} \mathbf{x} \to \mathbf{x_T} \qquad (16)$$

Proof of the above proposition is carried out using the LaSalle invariance principle [30].

Proof: Let Ξ be the Liapunov function candidate:

$$\Xi = V(x) \qquad (17)$$

Its time derivative is: $\dot{\Xi} = \nabla V(x)^T \dot{x}$. (18)

Substituting: $\dot{x} = -\nabla V(x)$ (19)

in (18) yields: $\dot{\Xi} = -\|\nabla V(x)\|^2$ (20)

∇V will vanish at the target point (x_T) and may have isolated critical points $\{x_i\}$ in Ω. This results in:

$$\dot{\Xi} \le 0 \qquad \forall \quad x\,, \qquad (21)$$

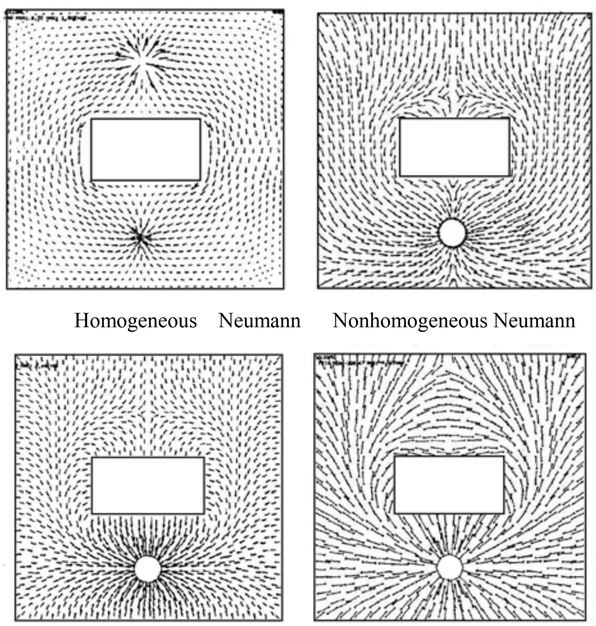
According to LaSalle principle any bounded solution of (19) will converge to the minimum invariant set:

$$E \subset \{x_T \cup \{x_i\}\} \quad . \qquad (22)$$

Determining E requires studying the critical points of V(x) where $\nabla V(x)=0$. According to the maximum principle, x_T is the only minimum (stable equilibrium point) V(x) can have. Besides x_T, V(x) has other critical points $\{x_i\}$ at which $\nabla V=0$; however, the hessian at these points is non-singular, i.e. V(x) is Morse. From the above we conclude that E contains only one point which is the point $x = x_T$ to which motion will converge.

IV. Results:

There are several settings in which a harmonic potential field may be configured for navigation. These configurations depend on the boundary conditions that are used to factor the crisp environment that is described using appropriate geometric functions in the process producing the navigation control policy. Some of these settings are discussed in [31,32]. Each one of these configurations possess distinct topological properties that are reflected in the integral and differential properties of the generated path. Figure-8 shows the control navigation policy of four different configurations for a simple rectangular environment.



Homogeneous Neumann Nonhomogeneous Neumann

Dirichlet Biharmonic

Figure-8: Navigation policies, different settings, HPF approach [32].

Here the suggested approach is simulated for a similar environment. The environment is factored into the navigation process using the probabilistic descriptor (pseudo PDF) of terrain suitability for motion. The descriptor is equal to zero inside the obstacle and one in the workspace.

The navigation control policy and the generated path are shown in figure-9. Unlike the deterministic case where the navigation policy degenerates inside the region to be avoided, the probabilistic HPF approach maintains the navigation field inside this region. This is of practical value since if a disturbance occur throwing the robot inside a forbidden region it can resume motion to get out of it instead of staying motionless in that region. The generated path is well-behaved with reasonable length compared to the optimum. It has almost constant curvature and keeps a healthy distance away from the forbidden regions.

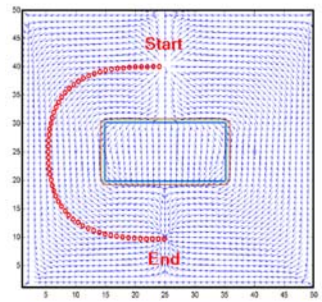


Figure-9: path and navigation policy - suggested approach

In figure-10 a more challenging environment is used to test the approach. The probabilistic descriptor is shown as an intensity image where the brighter is the area, the more fit it is for navigation. As can be seen segmenting this map into regions suitable for navigation and others that are not is very difficult if at all possible. Moreover a binary segmentation could result in creating isolated regions that are disconnected from the rest of the workspace. The generated path superimposed on the image of the probabilistic descriptor of the environment is also shown in figure-10. Visually assessing the path, it can be seen that the path is smooth, has a reasonable length and is practically restricted to the bright areas that are best suited for navigation. Figure-11 shows another nondivisible environment with paths generated by the planner for different start and end points.

V. Conclusion:

In this paper the capabilities of the HPF approach are extended to tackle planning in environments with inherent uncertainty that denies an operator the ability to segment the workspace into geometric regions of homogeneous attributes. The approach has several practical advantages. It makes it possible to directly use the sensor ambiguity map in planning a path with good properties without having to process the data and interpret it in a deterministic manner. Incorporating the environment in the action generation process using a task-centered, probabilistic field makes it possible to use vague representations of an environment to generate actions in a provably-correct manner. The suggested planner is expected to be useful, among other things, in building practical, reliable, low-cost navigation systems.

Acknowledgment: The author acknowledges KFUPM support.

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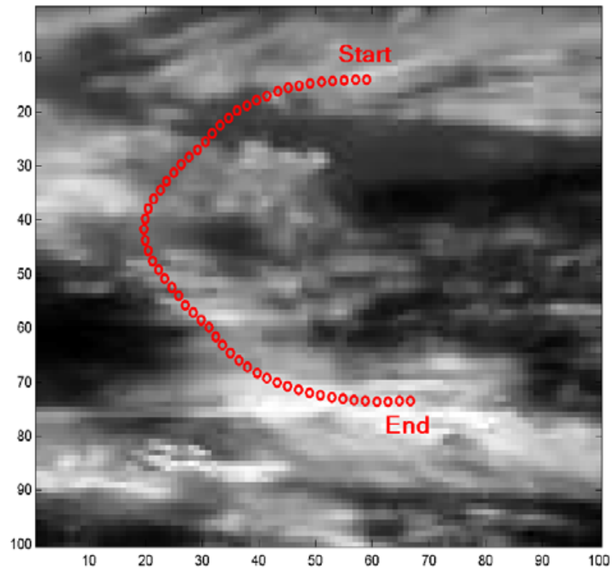

Figure-10: generated path

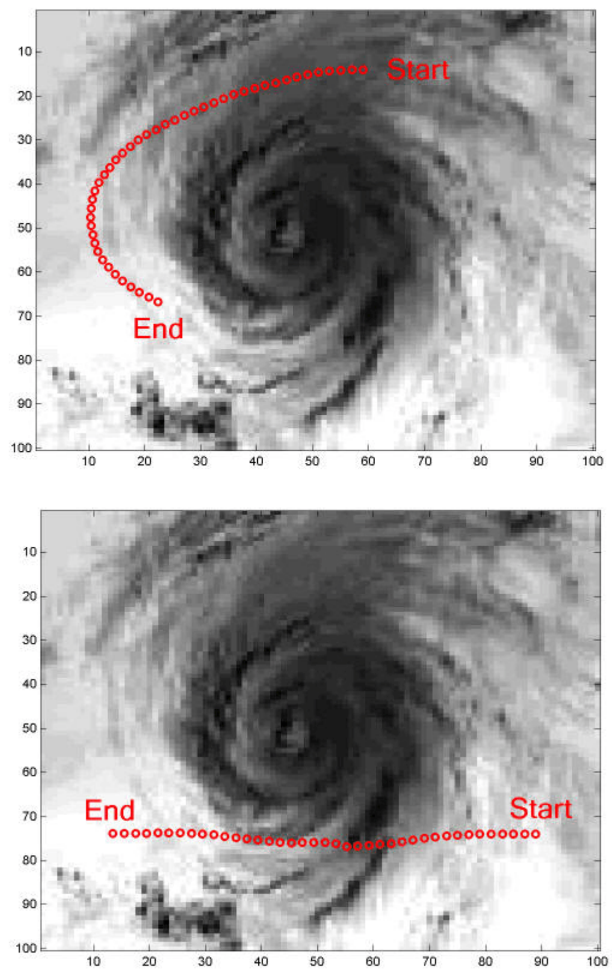

Figure-11: different paths, same environment.