Electric Circuits II

More about Mutual Inductance

Lecture #32

The material to be covered in this lecture is as follows:

- Mutual Inductance in Terms of Self Inductance
- Mesh Current Analysis of magnetically coupled Circuits
- Procedure for Determining Dot Markings
- Energy Calculation

After finishing this lecture you should be able to:

- Analyze Circuits with Mutual Inductance
- Determine the Coefficient of Coupling
- Determine the Dot Markings of neighboring Coils
- Perform Energy Calculations in Magnetically Coupled Circuits

Mutual Inductance in Terms of Self Inductance

We have seen that

$$L_1 = N_1^2 p_1$$

$$L_2 = N_2^2 p_2$$

Therefore $$L_1 L_2 = N_1^2 N_2^2 p_1 p_2 = N_1^2 N_2^2 (p_{11} p_{21})(p_{22} p_{12})$$

But for linear system,

$$p_{21} = p_{12} \Rightarrow$$

$$L_1 L_2 = N_1^2 N_2^2 p_1^2 \left(1 + \frac{p_{11}}{p_{12}}\right)\left(1 + \frac{p_{22}}{p_{12}}\right)$$

$$= M^2 \left(1 + \frac{p_{11}}{p_{12}}\right)\left(1 + \frac{p_{22}}{p_{12}}\right)$$

Replacing the positive quantity $\left(1 + \frac{p_{11}}{p_{12}}\right)\left(1 + \frac{p_{22}}{p_{12}}\right)$ by $\frac{1}{k^2}$ gives a more meaningful expression

$M^2 = k^2 L_1 L_2$ or $M = k\sqrt{L_1 L_2}$

Mutual Inductance in Terms of Self Inductance (Cont.)

k is called the Coefficient of Coupling and must lie between 0 and 1:

$0 \le k \le 1$

When $\Phi_{12} = \Phi_{12} = 0 \Rightarrow p_{12} = 0$

$$\Rightarrow \frac{1}{k^2} = \infty, \text{or } k = 0$$

If no flux linkage between the coils, obviously $M = 0$ and $k = 1$, when $\Phi_{11} = \Phi_{22} = 0$, which presents the Ideal State.

Mesh Current Analysis

- To solve the following example 32-1, Let us reproduce the Mesh Current Equations that describe the following Circuit seen in Lec31:

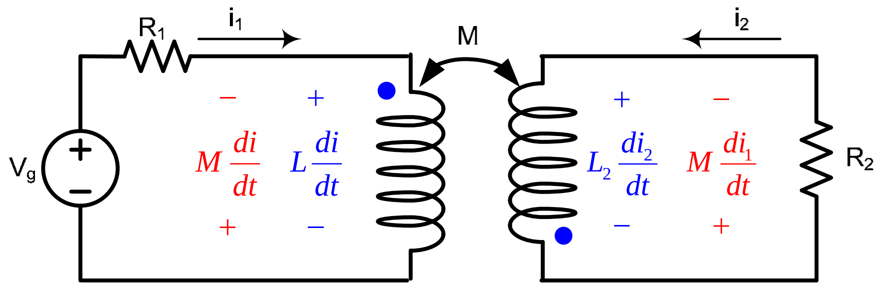

Fig.31-5 (c) The Self and Mutually Induced Voltages appearing across the coils shown in Fig.31-5 (a)

$$-v_g + R_1 i_1 + L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = 0 \quad \text{(Mesh 1)}$$

$$R_2 i_2 + L_2 \frac{di_2}{dt} - M \frac{di_1}{dt} = 0 \quad \text{(Mesh 2)}$$

Example 32-1

- Write a Set of Mesh-Current Equations that describe the Circuit in the following Figure:
- What is the Coefficient of Coupling of the Magnetically Coupled Coils?

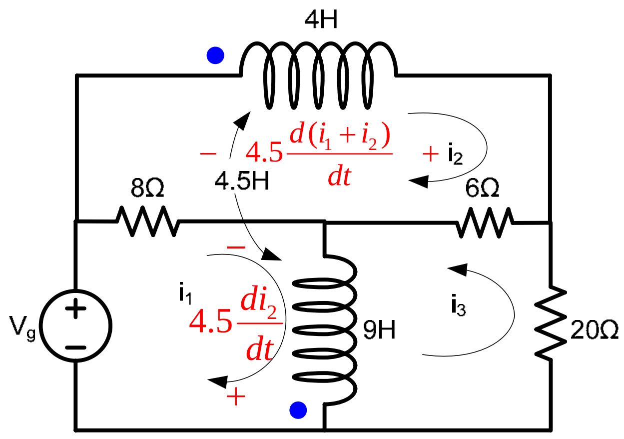

Fig.32-1 Circuit for Example 32-1

$$-v_g + 8(i_1 - i_2) + 9\frac{d(i_1+i_3)}{dt} - 4.5\frac{di_2}{dt} = 0 \quad \text{(Mesh 1)}$$

$$6(i_2 + i_3) + 8(i_2 - i_1) + 4\frac{di_2}{dt} - 4.5\frac{d(i_2+i_3)}{dt} = 0 \quad \text{(Mesh 2)}$$

$$20i_3 + 6(i_2 + i_3) + 4\frac{di_2}{dt} + 9\frac{d(i_1+i_3)}{dt} - 4.5\frac{di_2}{dt} = 0 \quad \text{(Mesh 3)}$$

$$k = \frac{M}{\sqrt{L_1L_2}} = \frac{4.5}{\sqrt{4\times 9}} = \frac{4.5}{6} = 0.75$$

Procedure for Determining Dot Markings

The following six steps applied to the following figure determine a set of Dot Markings.

1) Arbitrarily Select one Terminal, (Say Terminal D) of one coil and mark it with a Dot
2) Assign a current into the dotted terminal and label it i_D
3) Use the Right–Hand rule to determine the direction of the magnetic field established by i_D inside the coupled coils and label this as Φ_D
4) Arbitrarily pick one terminal of the second coil (Say Terminal A) and Assign a current into it (Show this as i_A)
5) Use the Right–Hand rule to determine the direction of the magnetic field established by i_D inside the coupled coils and label this as Φ_A
6) Compare the directions of the two fluxes Φ_D and Φ_A. If the fluxes have the same reference direction, place a Dot on the terminal of the second coil where the test current (i_A) enters.

In this example the fluxes Φ_D and Φ_A have the same reference direction, therefore a Dot goes on terminal A.

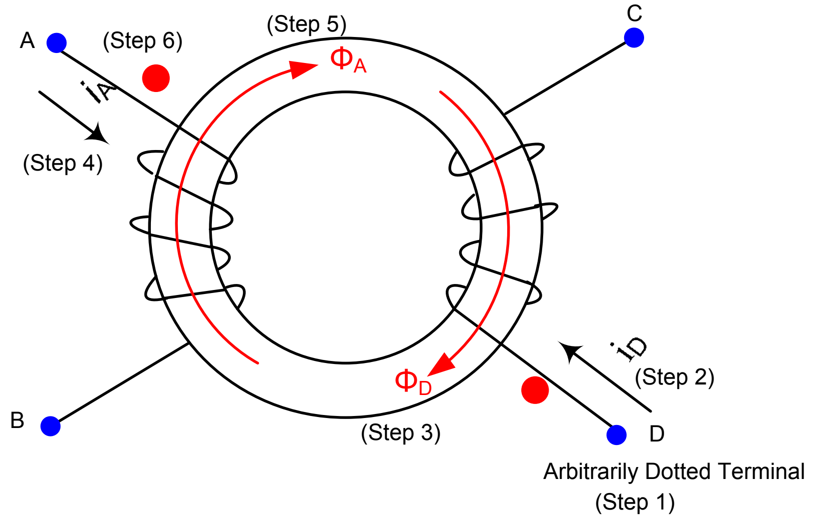

Fig.32-2 Set of Coils Showing Steps for Determining a Set of Dot Markings

Example 32-2

- The polarity markings on two coils are to be determined experimentally. The experimental setup is shown in Fig.32-3. When the switch is opened, the dc Voltmeter kicks upscale. According to the polarity and dot marking shown, where should we place the other dot?

Answer:

- ✓ The lower terminal of the unmarked coil has the same instantaneous polarity as the dotted terminal.
- ✓ Therefore, place a dot on the lower terminal of the unmarked coil.

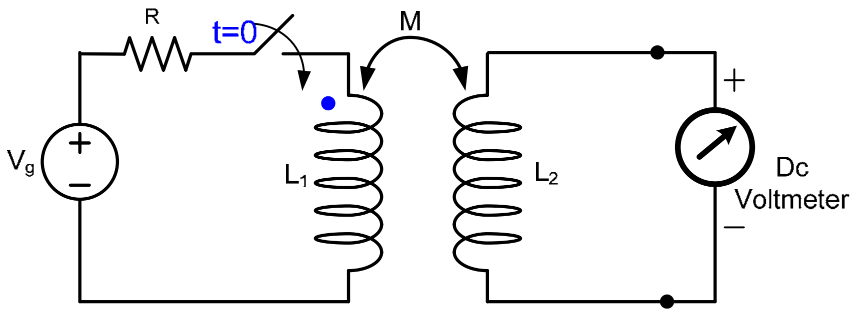

Fig.32-3 (a) Circuit For Example 32-2

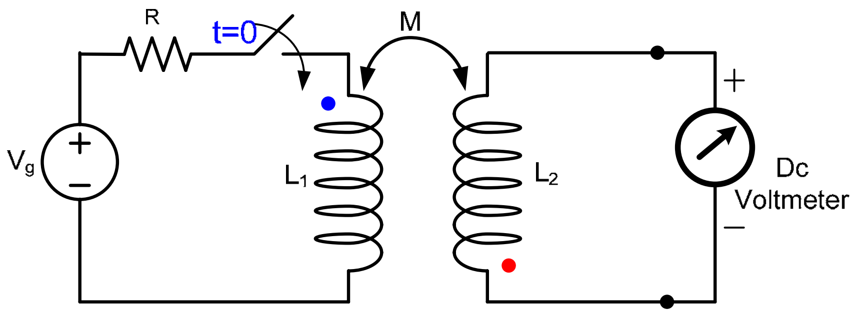

Fig.32-3 (b) Circuit For Example 32-2

Energy Calculations

For linear magnetic coupling

① $M_{12} = M_{21}$

② $M = k\sqrt{L_1 L_2}$

Consider the following Circuit

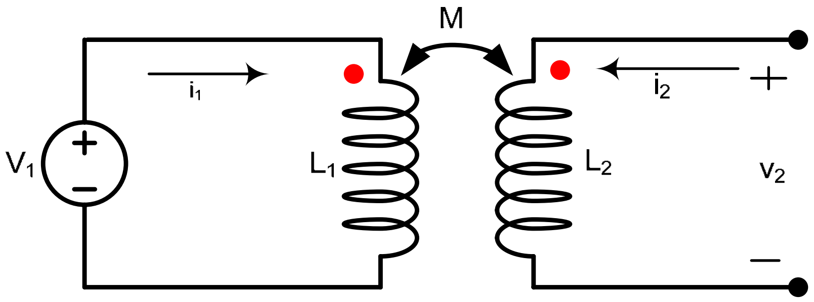

Fig.32-4 Circuit used for Energy Calculations

Case 1:

Increase i_1 from 0 to I_1, hold $i_1 = I_1$ and $i_2 = 0$

$$\int_0^{W_1} dw = L_1 \int_0^{I_1} di_1$$

$$W_1 = \frac{1}{2} L_1 I_1^2$$

Case 2:

Now we hold $i_1 = I_1$ and Increase i_2 from 0 to I_2

$$p = I_1 M_{12} \frac{di_2}{dt} + i_2 v_2$$

Energy Calculations (cont)

The total energy stored in the pair of coils when $i_2 = I_2$

$$\int_{W_1}^{W} dw = \int_0^{I_2} I_1 M_{12} di_2 + \int_0^{I_2} L_2 i_2 di_2$$

$$W = W_1 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$

$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{12} I_1 I_2$$

If we reverse the procedure – That is, if we first increase i_2 from zero to I_2 and increase i_1 from zero to I_1- The total energy stored is:

$$W = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M_{21} I_1 I_2$$

At any instant of time, the total energy stored in the coupled coils is:

$$w(t) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2$$

Assuming that both coil currents entered polarity marked terminals

In general:

$$w(t) = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 \pm M i_1 i_2$$

Example 32-3

Consider the circuit in Fig.32-5. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time $t = 1s$ if $v = 60\cos(4t + 30^\circ)V$.

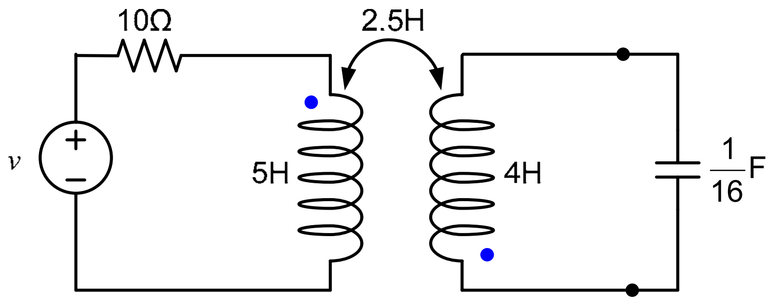

Fig.32-5 Circuit For Example 32-3

Solution:

The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

- Means that the inductors are tightly coupled.
- To find the energy stored, we need to obtain the frequency domain equivalent of the circuit

$$60\cos(4t + 30^\circ)V \Rightarrow 60\angle 30^\circ, \omega = 4\,{rad}/{s}$$

$$5H \Rightarrow j\omega L_1 = j\,20\Omega$$

$$2.5H \Rightarrow j\omega M = j\,10\Omega$$

$$4H \Rightarrow j\omega L_2 = j\,16\Omega$$

$$\frac{1}{16}F \Rightarrow \frac{1}{j\omega C} = -j\,4\Omega$$

The frequency domain equivalent is shown in Fig.32-6 below:

Solution (Cont):

We now apply mesh analysis

$$(10+j20)I_1+j10I_2=60\angle 30^\circ$$

$$j10I_1+(j16-j4)I_2=0$$

Or

$$I_1=-1.2I_2$$

This yields

$$I_2(-12-j14)=60\angle 30^\circ \Rightarrow I_2=3.254\angle 160.6^\circ\,A$$

$$I_1=-1.2I_2=3.905\angle -19.4^\circ\,A$$

In the time domain

$$i_1=3.905\cos\left(4t-19.4^\circ\right),$$

$$i_2=3.254\cos\left(4t+160.6^\circ\right)$$

At time t=1s, 4t= 4rad=229.2º,

$$i_1=3.905\cos\left(229.2^\circ-19.4^\circ\right)=-3.389A,$$

$$i_2=3.254\cos\left(229.2^\circ+160.6^\circ\right)=2.824A$$

Solution (Cont):

The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$

$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73J$$

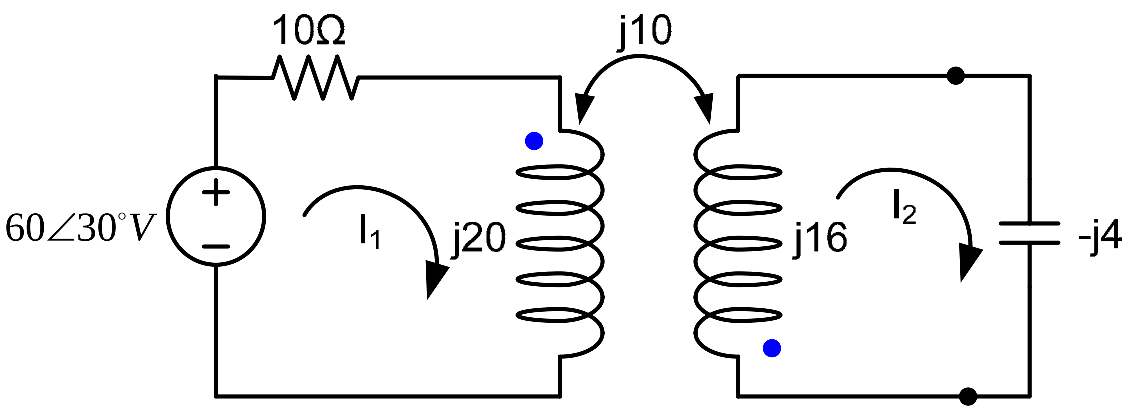

Fig.32-6 Frequency domain Equivalent Circuit For Example 32-3

Self Test:

- When *k* equals zero, two inductors in series combine by
  a) addition   b) subtraction   c) product-over-sum   d) multiplication
  answer: a
- Two parallel inductors with no mutual inductance combine by
  a) addition   b) subtraction   c) product-over-sum   d) multiplication
  answer: c
- If N=200 and a rate of change of flux is 0.4 Wb/s. The voltage induced in the inductor is?
  a) 50   b) 80   c) 200   d) 500
  answer: b
- Tow inductors are wound on a ferromagnetic core. The first Inductor has N_1=200 and a rate of change of flux is 0.4 Wb/s. If the second Inductor has N_2=125 turns and the coefficient of coupling is unity, how much voltage is induced in the second inductor?
  a) 50   b) 80   c) 200   d) 500
  answer: a
- The self-inductances of the coils in Fig.32-4 are L_1= 18 mH and L_2 = 32 mH. If the coefficient of coupling is 0.85, calculate the energy stored in the system in millijoules when
  a) $i_1 = 6$ A, $i_2 = 9$ A;   b) $i_1 = -6$ A, $i_2 = -9$ A;   c) $i_1 = -6$ A, $i_2 = 9$ A;   d) $i_1 = 6$ A, $i_2 = -9$ A.
  What conclusion do you draw?
  answer:
  a) 2721.60 mJ;   b) 2721.60 mJ;   c) 518.40 mJ;   d) 518.40 mJ.
  Same answer when both current are entering or both leaving, and same answer when one is leaving and one is entering.