Generalized Balanced and Approximately Balanced Representations

by

Mohammed Ahmed Ibrahim Ubaid

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

ELECTRICAL ENGINEERING

July, 1992

Order Number 1354078

Generalized balanced and approximately balanced representations

Ubaid, Mohammed Ahmed Ibrahim, M.S.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1992

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

GENERALIZED BALANCED AND APPROXIMATELY BALANCED REPRESENTATIONS

BY

MOHAMMED AHMED IBRAHIM UBAID

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

In

ELECTRICAL ENGINEERING

JULY 1992

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN, SAUDI ARABIA
COLLEGE OF GRADUATE STUDIES

This thesis, written by ***MOHAMMAD AHMAD IBRAHIM UBAID*** under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirement of the degree of of ***MASTER OF SCIENCE in ELECTRICAL ENGINEERING.***

THESIS COMMITTEE

Dr. U. M. Al-Saggaf (Advisor)

Dr. M. Bettayeb (Member)

Dr. T. Bakri (Member)

Dr. J. Bakhashawain (Member)

Department Chairman

Dean, College of Graduate Studies

ACKNOWLEDGMENT

First of all, praise be to "ALLAH" whose help made it possible to complete this work.

Acknowledgment is due to King Fahd University of Petroleum and minerals for support of this research.

I wish to thank my thesis advisor Dr. Ubaid Al-Saggaf for his patience and careful guidance during the course of work. I am very grateful to my other committee members Dr. Maamar Bettayeb, Dr. Talal Bakri and Dr. Jamil Bakhashawain for their helpful remarks and cooperation.

I would also like to thank my friends and colleagues especially in electrical engineering department for their encouragement and support.

I am indebted to all my family: my mother, father, brothers, sisters, wife and my little daughters for their love, inspiration and support.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ملخص الرسالة :

اسم الطالب : محمد احمد ابراهيم عبيد

عنوان الرسالة : تعميم التمثيلات المتوازنة وتقريبها

التخصص : هندسة كهربائية

تاريخ الشهادة : محرم ١٤١٣هـ الموافق يوليو ١٩٩٢م

هذه الرسالة تتكون من ثلاثة أجزاء رئيسية :

في الجزء الأول : قمنا بتخفيف الشروط على معادلات ليابنوف المصفوفية المعممة والناتجة عن نظرية تجمع الجذور بالنسبة للأنظمة التي يمكن التحكم فيها ، وقد أستخدمت هذه النتيجة في طرق تصغير النماذج المتوازنة لإنتاج نموذج مصغر بحيث تكون جذوره في نفس منطقة تجمع جذور النموذج الأصلي .

في الجزء الثاني : عندما يستخدم التحويل الثنائي الخطي فإن حل معادلة التحكمية لنظام ما يتساوى مع حل معادلة التحكمية المعممة لتحويل هذا النظام بالنسبة لدائرة معرفة . وبدمج التحويل الثنائي الخطي مع النظرية السابقة طورنا طريقة تصغير النماذج المتوازنة ، وبهذا التطوير نستيع جعل الاستجابة الترددية للخطأ قليلة في منطقة الترددات الصغيرة .

أما في الجزء الثالث : فإن التعقيدات المصاحبة للأنظمة ذات الأحجام الكبيرة والناتجة عن الحسابات اللازمة لإيجاد نموذج مصغر بإستخدام طرق تصغير النماذج المتوازنة تجعل تلك الطرق غير عملية . ومع ذلك فان من الطبيعي ان يوجد ارتباط ضعيف بين اجزاء النظم الكبيرة ، فهذا يساعد على إيجاد انظمة مصغرة تقريبية لتلك الأنظمة الكبيرة بعمليات حسابية أقل وموثوق بها . وأخيراً أوجدنا الشروط لصحة هذا التقريب ، وكذلك اوجدنا حداً أعلى للتقريب .

درجة الماجستير في العلوم

جامعة الملك فهد للبترول والمعادن

الظهران ـ المملكة العربية السعودية

محرم ١٤١٣هـ الموافق يوليو ١٩٩٢م

ABSTRACT

Title : **GENERALIZED BALANCED AND APPROXIMATELY BALANCED REPRESENTATIONS**

By : **Mohammed Ahmed Ibrahim Ubaid**

Major Field : **Electrical Engineering**

Date : **July 1992**

This thesis consists of three main parts. In the first part, the conditions on the generalized Lyapunov matrix equations that are produced from poles clustering theorem is relaxed for controllable system. This result is used in balanced model reduction to produce a reduced order model with poles clustered in the same region as the poles of the full order model.

In the second part, when bilinear transformation is used the controllability gramian of the system is equal to the generalized controllability gramian of the transformed system with respect to a given circle. This circle is determined from the bilinear transformation. Using bilinear transformation and the above theorem, a new balanced model reduction technique is developed. In this technique, the error frequency response is forced to be a high pass instead of low pass.

In the third part, the complexity of a large scale system makes the computations of reduced order models based on balancing impractical. However, typically there is a weak coupling between the subsystems of a large scale system. This is used to derive approximate balanced-truncated reduced order models of a large scale discrete system with reliable and tractable computations. The condition for validity of the approximations and bounds on the norms of the approximation errors are, also, derived.

MASTER OF SCIENCE DEGREE
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN, SAUDI ARABIA

CHAPTER 1

INTRODUCTION

1.1 General

Mathematical modelling of natural phenomenon leads to coupled high order differential or partial differential equations and requires very high dimensional representations. Computational costs and components needed for analysis, testing or implementing of such systems are quite excessive. Therefore, model reduction methods are needed to reduce the order of such system to a manageable size. Some of these methods are computationally simple such as Pade' approximation [50], modal approximation [24] and continued fraction expansions [14]. However, they generally have no guaranteed performance. Other methods depend on the minimization of some performance indices [43],[58],[59]. These methods are computationally demanding and suffer from many problems such as the choice of starting guesses, convergence and multiple local minima.

Recently, a method for model reduction based on measure of controllability and observability was suggested by Moore [44]. This method has dramatically changed the status of model reduction. In this method, a representation, termed balanced, is chosen for the system such that the input-to-state coupling and the state-to-output coupling are weighted equally so that those state components which are weekly coupled to both input and output are discarded (Moore [44]). This method is computationally simple requiring

only standard matrix software. Moreover, it is possible to predict the error between the frequency response of the full order model and the reduced order model (Enns [16], Glover [23], Al-Saggaf and Franklin [7]). Furthermore, if the full order model is stable, controllable and observable, the reduced model will generically be stable, controllable and observable [46].

Despite all the properties of the balanced-truncation reduced order model, there is no guarantee that the poles of the reduced order model will be clustered in the same region as the poles of the full order model. The locations of the poles affect the time response characteristics such as overshoots, oscillations, settling time, etc... For the reduced order model to be a faithful representation of the full order model, the least expected is that the poles of the full and reduced order models lie in the same region. This will often ensure that the reduced order model has comparable response characteristics as the full order model. Moreover, most of the methods for model reduction are based on assumption that the original system has constant parameters. However, it is shown in [30] that even if the reduced order model is stable and very accurate approximation of the full order model, the closed-loop system characteristics may not be acceptable, and the stability of the closed-loop system may not even be preserved due to the effect of parameters variations. Typically in practical cases, parameters variations will result in movements of the system poles from their nominal values. If the system poles for all parameter variations are clustered in a certain region and the reduced order model has its poles in this region then it will be a faithful representation not only for the nominal model but also for all models resulted from parameter variations within that region.

Balancing provides an effective and a numerically economical way to do model reduction. However, sometimes it is important that the reduced order model has a small reduction error at a certain frequency band, especially at low frequency band. This is also important when using the reduced order model in feedback control system design. There, an accurate approximation of the full order model is needed at the crossover frequency region. Moreover, the balanced realization techniques give good approximations of the impulse response but have a drawback of giving a large steady state error for step inputs. This indicates that the approximation is better for high frequency region than low frequency region. For control system design, however, the low frequencies behavior is more important than high frequencies ones and hence a good approximation at low frequencies is highly important.

Another problem in balanced representation is the determination of the representation for the case of large scale systems (say order 1000 or more). It is still a challenging open problem [39]. Moreover, balancing is intrinsically badly conditioned for system with states nearly uncontrollable and/or unobservable. Since one of the main uses of balanced representations is to obtain reduced order models, methods were proposed to drive the reduced order models directly without computing the balancing transformation [48]. However, this does not eliminate the numerical difficulties associated with large scale systems since the solutions of two Lyapunov equations are still needed to derive the reduced order models. Nevertheless, in many situations, experience and intuition indicate how to split a large scale system into a set of simpler weakly coupled subsystems for the purpose of analysis and design. However, the approach that completely neglects coupling in large scale systems usually leads to unsatisfactory results. Al-Saggaf [2] used ε-coupling (weakly coupling) for large scale systems to define the controllability and observ-

ability of continuous-time systems as a power series in the coupling parameter ε. Then, a compromise is made between the numerical practicability of an approximate balanced-truncated reduced order model and how far it is from an exact balanced-truncation reduced order model.

1.2 Literature Review

The model reduction problem has been a major attraction in system theory literature and considerable attention has been devoted to it in the last few decades and one has only to examine the comprehensive list of references compiled by Genesio and Milanese [22] to appreciate this fact. Various reduction methodologies have been proposed and algorithms of diverse computational complexity have been presented. There are time domain and frequency domain methods. The most popular time domain reduction methods are aggregation, singular and regular perturbation, and balancing. Among the frequency domain reduction methods are: Pade' approximation, Routh approximation, Pade'-Routh, and moment matching, see Al-Saggaf and Bettayeb [64] for a review on model reduction techniques. In this section, we will be concerned with balanced model reduction. The following subsections give a literature review of balanced model reduction and its extensions.

1.2.1 Balanced Model Reduction

Balanced model reduction was first introduced by Moore [44]. Mullis and Roberts [45], while synthesizing minimum round off noise fixed point digital filters, obtained a system very similar to a balanced system. Pernebo and Silverman [46] studied model reduction of systems based on balanced realization as well as the stability of the reduced

order models. Bettayeb et al. [11], [53] further studied balancing. Silverman and Bettayeb [10] studied discrete balancing. Bettayeb and Djennoune in [12] obtained a bound for closeness of the eigenvalues of the subsystems and those of the full order system. Also, Al-Saggaf studied discrete balancing and obtained a relation between the eigenvalues of the subsystems and the full order model for discrete time case [5]. Fernando and Nicholson [17-20] further studied balancing and proposed using the slow subsystem of the balanced system as the reduced order model. This is particularly useful in discrete time systems because the reduced order system will also be balanced which is not the case if the usual reduction method is followed. However, Al-Saggaf and Franklin [7] proposed a method for obtaining a balanced reduced order model, not necessarily the slow subsystem, for discrete time systems. Shokoohi et al. [51,52] have considered balancing linear time-variable systems and Verriest et al. [57] have considered balancing the general class of analytic time-varying linear systems. Balanced realization of singularly perturbed systems have been studied by Shahruz and Behtash in [49] and by Bettayeb and Djennoune in [13]. Also, Liu and Anderson [41] obtained a reduced order system after approximating the balanced system by singularly perturbed form.

There are several algorithms for computing the balancing transformation. One efficient algorithm is due to Laub et. al.[39]. Safonov and Chiang [48] developed a reliable algorithm for getting the reduced order model without computing a balancing transformation. This transformation is intrinsically badly conditioned for systems with some nearly uncontrollable and/or unobservable modes. Hence, this algorithm solves these numerical problems associated with computing the balancing transformation. Moreover, this algorithm does not require the system (A,B,C) to be controllable and observable.

For unstable systems, Kenney and Hewer [36] gave necessary and sufficient condition for balancing unstable system. Therapos [55] developed a method for balancing transformation of unstable nonminimal linear system. Al-Saggaf [3] developed generalized normal representation for which the results in [36] and [55] are special cases. The reduction method in [3] results in reduced order models with the same number of unstable poles as the full order model and with an apriori upper bound on the reduction error which is very important when using the reduced order models in feedback control system design.

1.2.2 Frequency Weighted Balanced Model Reduction

Enns [16] is the first to propose a model reduction method based on frequency weighting and balanced realization. For either input-weighting or output-weighting, his method yields stable reduced order model; however, when both input-weighting and output-weighting are included his method fails to guarantee stability of the reduced order model. Also, no error bound for his method is k nown. Al-Saggaf and Franklin [7] developed frequency weighting techniques for continuous and discrete systems that extends the method of balancing and gave Chebyshev norm error bounds for their method. An important distinct feature of the method in [7] that is different from other frequency weighted model reduction techniques is that the frequency weighting need not to be stable and the reduction error has transmission zeros at the poles of the frequency weighting. Lin and Chiu [40] proposed another weighted frequency balanced realization. Their method requires a stable weighting and yields stable reduced order model.

1.2.3 Approximately Balanced Representation

Jonckheere and Silverman [33] showed that for deformable systems, under some assumptions, balanced-truncation and optimal Hankel-norm approximation are equivalent to model truncation in asymptotic sense as the damping ratio is reduce to zero. Jonckheere [32] and Jonckheere and Opendenacker [34] used a parameterization of balanced SISO systems to show the same result. These were shown for a more general system by Gregory [26] and then even more general systems by Belloch et al. [9]. However, all the above results were specialized to flexible systems with the damping ratio approaching zero. Al-Saggaf [2] derived approximate balanced-truncation reduced order model for continuous-time weakly coupled large scale systems with reliable and tractable computations.

1.3 Thesis Contributions

* Relaxation of poles clustering theorem for controllable systems.

* A new balanced reduction method is presented. This method produces a reduced order model with poles clustered in the same region as the poles of the full order model.

* It is proven that the controllability (observability) gramian does not change under bilinear transformation if we use a generalized controllability (observability) gramian for the transformed system instead of the normal one.

* Another new reduction technique is developed. In this reduction method we can force the error frequency response to be high pass instead of being low pass, using bilinear transformation. This will give good approximation of the full order model at low frequencies.

* Derivation of approximate balanced-truncation reduced order models for discrete-time weakly-coupled large scale systems with reliable and tractable computations.

1.4 Thesis Organization

The remainder of this thesis is organized into four chapters. Chapter 2 will cover the generalized model reduction. In chapter 3, model reduction using balanced realization with improved frequency behavior will be presented. Chapter 4 will cover the approximate balanced-truncation model reduction for weakly coupled systems and finally chapter 5 will give the summary and recommendations for further research.

In chapter 2, balanced model reduction for continuous-time and discrete-time systems and their properties will be reviewed. Then, poles clustering theorem will be used to produce the generalized Lyapunov equation for the desired regions. This theorem will, also, be relaxed for controllable systems. The modification of balanced model reduction will be developed. Finally, examples will be used to compare balanced model reduction and the generalized model reduction.

In chapter 3, bilinear transformation and its properties will be reviewed. Then, it will be prove that the controllability (observability) gramian of a system is equal the the generalized controllability (observability) gramian of the transformation of the system using a bilinear transformation. The proposed technique, that produces reduced order

models with samll error frequency response at low frequency region, and some of its properties are presented. Finally, the proposed technique is illustrated by numerical examples.

In chapter 4, weekly coupled (ε-coupling) systems are defined and approximations for the controllability and observability gramians of continuous-time systems will be introduced. The condition for validity of these approximations is given and bounds on the norms of the approximation errors are derived. Also, a measure of the closeness of the approximately balanced to the actual one is given. The approximations for the controllability and observability gramians of discrete-time systems, the condition for the validity of these approximations and bounds on the norms of the approximation errors are developed. Finally, examples are used to illustrate the techniques developed in this chapter.

CHAPTER 2

GENERALIZED BALANCED MODEL REDUCTION

In this chapter, balanced model reduction will be modified to produce a reduced order model with poles clustered in the same region as the poles of the full order model. To do this, a generalized Lyapunov equation will be used instead of the normal Lyapunov equation. The generalized Lyapunov equation guarantees pole clustering in the desired region.

Balanced model reduction for continuous-time and discrete-time systems and their properties will be reviewed in section 2.1. In section 2.2, pole clustering theorem will be used to produce the generalized Lyapunov equation for the desired regions. This theorem will also be relaxed for controllable systems. In section 2.3, the modification of balanced model reduction will be developed and will be called *generalized balanced model reduction*. Finally, examples will be used to compare the balanced model reduction and the generalized model reduction.

2.1 Balanced Model Reduction

Recently, extensive studies on balanced representations have been undertaken for the determination of reduced order models which contain only the most controllable and most observable states of the system. The main idea behind balancing is that there exists state space coordinates where the controllability and observability gramians are equal and diagonal. The diagonal entries of the gramians are called the Hankel singular values of the system and they provide a measure of *how much* a state is controllable and observable. A natural way to achieve model reduction is to keep only the most controllable and most observable states corresponding to the largest Hankel singular values of the system.

2.1.1 Continuous-Time systems

In this subsection, continuous-time balanced model reduction will be reviewed and the different characteristics of the balanced reduced order model will be summarized.

2.1.1.1 Reduced Order Model

Assume that the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1a}$$

$$y(t) = Cx(t) \tag{2.1b}$$

is asymptotically stable, controllable and observable where $A \in R^{n \times n}$, $B \in R^{n \times m}$, and $C \in R^{p \times n}$. The controllability gramian P and observability gramian Q are defined as

$$P = \int_0^\infty e^{At} BB' e^{A't} dt \tag{2.2}$$

$$Q = \int_0^\infty e^{A't} C'C e^{At} dt \tag{2.3}$$

and can be found as the unique positive definite solution of the following Lyapunov equations:

$$AP + PA' + BB' = 0 \tag{2.4}$$

$$A'Q + QA + C'C = 0 \tag{2.5}$$

where (') denotes the transpose. If the representation of the system of equation (2.1) is transformed to another representation using a non-singular transformation T, then, the new state space representation of the system is $(\overline{A},\overline{B},\overline{C})$ where

$$\overline{A} = T^{-1}AT,\ \overline{B} = T^{-1}B,\ C = CT$$

and the gramians will be transformed to

$$\overline{P} = T^{-1}PT^{-1},\ \overline{Q} = T'QT,\ \overline{PQ} = T^{-1}PQT$$

Thus, the gramians P and Q depend on the state-space coordinates. However, the eigenvalues of their products PQ are invariant under state space transformations and are input/output invariant. The square root of the eigenvalues of PQ are called the Hankel singular values of the system $G(s) = C(sI - A)^{-1}B$ and they are denoted by σ_i where $\sigma_i = \lambda_i^{\frac{1}{2}}(PQ)$ and $\lambda_i(PQ)$ is the iTh eigenvalue of PQ.

A realization $(\overline{A},\overline{B},\overline{C})$ of $G(s)$ is said to be balanced, if $\overline{P} = \overline{Q} = \Sigma$ where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. The states of a balanced representation are balanced between controllability and observability. Thus, they represent a convenient structure for model reduction since those states having week controllability and week observability can be neglected without causing any imbalance in controllability or in

observability properties of the remaining states [44].

Let the balanced system be partitioned as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}B_1\\ B_2\end{pmatrix}u \tag{2.6a}$$

$$y=(C_1 \quad C_2)\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{2.6b}$$

where $x_1 \in R^r$ contains the most controllable and most observable states and $x_2 \in R^{n-r}$ contains the least controllable and least observable states. Also, let Σ be partitioned compatibly as

$$\Sigma=\begin{pmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{pmatrix} \tag{2.7}$$

where $\Sigma_1 = diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ and $\Sigma_2 = diag(\sigma_{r+1}, \ldots, \sigma_n)$. If $\frac{\sigma_r}{\sigma_{r+1}} \gg 1$, then the subsystem

$$\dot{\bar{x}}_1 = A_{11}\bar{x}_1 + B_1 u \tag{2.8a}$$

$$y_r = C_1\bar{x}_1 \tag{2.8b}$$

is taken as the reduced order model of the full order balanced system and will contain the most controllable and most observable parts of the system [44].

2.1.1.2 Properties of Balanced Systems

In the following, properties of the continuous-time balanced system and the properties of reduced order model are reviewed.

Theorem 2.1 [46]

Assume that Σ_1 and Σ_2 have no diagonal entries in common. Then, both subsystems (A_{ii}, B_i, C_i) $(i = 1,2)$ of the balanced system are asymptotically stable and internally balanced with gramians Σ_i (i=1,2).

Theorem 2.2 [16,23]

Let (A,B,C) be a balanced continuous-time system, then the following gives an upper bounds for reduction error in the balanced scheme

$$\| G(jw) - G_r(jw) \|_\infty \le 2(\sigma_{r+1} + \ldots\ldots + \sigma_n) = 2tr(\Sigma_2) \tag{2.9}$$

where $G(s) = C(sI - A)^{-1}B$, $G_r(s) = C_1(sI - A_{11})^{-1}B_1$, tr denotes the trace and infinity norm is defined as

$$\| X(s) \|_\infty = \sup_{w \ge 0} \bar{\sigma}[X(jw)] \tag{2.10}$$

and $\bar{\sigma}(X)$ is the maximum singular value of X.

2.1.2 Discrete-Time Systems

A review of the discrete-time analogue of the results in the last subsection will be given here. Although, the analysis of discrete-time balanced systems is similar to that of continuous-time balanced systems, there are some major differences between the two.

Assume that the following discrete-time system is asymptotically stable, reachable and observable

$$x(k+1) = Ax(k) + Bu(k) \tag{2.11a}$$

$$y(k) = Cx(k) \tag{2.11b}$$

The reachability gramian P and observability gramian Q are defined as

$$P = \sum_{k=0}^{\infty} A^k BB'(A')^k \tag{2.12}$$

$$Q = \sum_{k=0}^{\infty} (A')^k C'CA^k \tag{2.13}$$

and can be found as the unique positive definite solution of the following Lyapunov equations:

$$APA' - P + BB' = 0 \tag{2.14}$$

$$A'QA - Q + C'C = 0 \tag{2.15}$$

As in the continuous-time case, it is possible to find a transformation matrix which makes the two gramians diagonal and equal to Σ. The rest of the analysis required to obtain the reduced order model is completely analogues to that of the continuous-time case; therefore, it will not be repeated here.

If the discrete-time system is balanced, then every subsystem is asymptotically stable as it is shown in the following theorem

Theorem 2.3 [46]

Assume the system (2.11) is asymptotically stable and that either the reachability or observability gramians is non-singular and diagonal. Then every subsystem is asymptotically stable.

Thus, for discrete-time systems, a stronger results for asymptotic stability of the subsystems is true than continuous-time systems. On the other hand, only weaker results are obtained for the reachability and observability of the subsystems as shown in the following theorem.

Theorem 2.4 [46]

Assume that the partitioning (2.6) is made for the discrete-time system (2.11) so that

$\sigma_r > \sigma_{r+1}$. Then, the subsystem (A_{11}, B_1, C_1) is reachable and observable.

Theorem 2.5 [4,6]

Let *(A,B,C)* be a balanced discrete-time system, then the following is an upper bounds for reduction error in the balanced scheme

$$\| G(e^{j\theta}) - G_r(e^{j\theta}) \|_\infty \leq 2(\sigma_{r+1} + \ldots + \sigma_n) = 2tr(\Sigma_2)$$

and strict inequality holds if $\sigma_i \neq \sigma_{i+1}$ for any i, $r \leq i \leq n-1$

2.1.3 Determination of the Transformation Matrix *T*

There are several algorithms for computing the balancing transformation. One efficient algorithm is due to Laub et al. (1987)[39]. In this algorithm the balancing transformation is computed as follows

(i) Compute P and Q

(ii) Compute Cholesky factors of the gramians

$$P = L_c L_c'; \quad Q = L_o L_o' \tag{2.15}$$

where L_c and L_o denotes the lower triangular Cholesky factors of P and Q respectively.

(iii) Compute singular value decomposition of the product of the Cholesky factor

$$L_o' L_c = U \Sigma V' \tag{2.16}$$

(iv) Form the balancing transformation and its inverse as follows

$$T = L_c V \Sigma^{-\frac{1}{2}}; \quad T^{-1} = \Sigma^{-\frac{1}{2}} U' L_o' \tag{2.17}$$

2.2 Pole Clustering Theorems

In stability studies of linear time-invariant systems one is often concerned with the poles location of the system. Most famous are regions of pole clustering with respect to the left complex plane and the unit circle. The notation of relative stability introduces more regions. In general, one may ask: given a matrix A and an algebraic region S in the complex plane, find the necessary and sufficient conditions for eigenvalues of A to lie in S. Important results on pole clustering can be found in [1],[15],[27-29],[35].

In this section, we are concerned with second order regions such as ellipses, parabolas, circles, etc,... These regions are described by

$$S = \{(x,y): \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy < 0\} \tag{2.18}$$

Let $A \in C^{n\times n}$, λ be an eigenvalue of A, $x = Re\lambda$, $y = Im\lambda$ and $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of A. The following theorem states the necessary and sufficient conditions for $\sigma(A) \in S$.

Theorem 2.5 [29],[35]

Let $A \in C^{n\times n}$ and consider S with $\gamma_{02} + \gamma_{20} \geq 0$. For $\sigma(A) \in S$, it is necessary and sufficient that given any positive definite Hermition matrix $M \in C^{n\times n}$, there exists a unique positive definite Hermition matrix $H \in C^{n\times n}$ such that

$$c_{00}H + c_{10}AH + c_{20}A^2H + c_{01}HA^* + c_{02}H(A^*)^2 + c_{11}AHA^* = -M \tag{2.19}$$

where

$$c_{00} = \gamma_{00}, \quad c_{01} = \bar{c}_{10} = \frac{1}{2}(\gamma_{10} + i\gamma_{01}), \; i = \sqrt{-1}$$

$$c_{11} = \frac{1}{2}(\gamma_{20} + \gamma_{02}), \quad c_{02} = \bar{c}_{20} = \frac{1}{4}[(\gamma_{20} - \gamma_{02}) + i\gamma_{11}]$$

and * denotes the complex-conjugate transpose.

In the Lyapunov equation for continuous- or discrete-time systems used for balanced model reduction, BB' or $C'C$ (which is equivalent to M in the above theorem) is positive semidefinite. Therefore, the above theorem needs to be generalized to positive semidefinite matrices. This can be done for controllable systems in the following theorem.

Theorem 2.6

Let $A \in C^{n\times n}$ and consider S with $\gamma_{02}+\gamma_{20} \geq 0$. For $\sigma(A) \in S$, it is necessary and sufficient that given any matrix, $B \in C^{n\times m}$ with (A,B) is controllable, there exists a unique positive definite Hermition matrix $H \in C^{n\times n}$ such that

$$c_{00}H + c_{10}AH + c_{20}A^2H + c_{01}HA^* + c_{02}H(A^*)^2 + c_{11}AHA^* = -BB^* \qquad (2.20)$$

where

$$c_{00} = \gamma_{00}, \quad c_{01} = \bar{c}_{10} = \frac{1}{2}(\gamma_{10} + i\gamma_{01})$$

$$c_{11} = \frac{1}{2}(\gamma_{20} + \gamma_{02}), \quad c_{02} = \bar{c}_{20} = \frac{1}{4}[(\gamma_{20} - \gamma_{02}) + i\gamma_{11}]$$

Moreover, if (2.20) still has positive definite solution H for B and $\lambda_i \notin S$, then λ_i belongs to the boundary of S.

Proof

Sufficiency : Let $A^*v = \bar{\lambda}v \Rightarrow v^*A = \lambda v^*$, $x = Re\lambda$ and $y = Im\lambda$. Multiplying (2.20) from left by v^*, from right by v and after simplifications we get

$$(\gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy)v^*Hv = -v^*BB^*v$$

since $v^*Hv > 0$ and $v^*BB^*v \geq 0$, this implies

$$\gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{20}y^2 + \gamma_{11}xy \leq 0$$

Since (A,B) is controllable, then $v^*B \neq 0$. Thus $v^*BB^*v \neq 0$; which implies

$$\gamma_{00}+\gamma_{10}x+\gamma_{20}x^2+\gamma_{01}y+\gamma_{02}y^2+\gamma_{11}xy<0 \Rightarrow \sigma(A)\in S$$

Necessity : Let $\lambda \in S$, we want to prove that for every B with *(A,B)* controllable there exists a positive definite solution H of (2.20). Since c_{11} can not be negative, the proof will be done for $c_{11}=0$ and $c_{11}>0$.

Let c_{11}=0, equation (2.20) becomes

$$K^*H+HK=-BB^*$$

where $$K=\frac{1}{2}c_{00}I+c_{10}A+c_{20}A^2$$

If v is an eigenvector of A, then it is also an eigenvector of K. Hence, *(A,B)* is controllable implies *(K,B)* is controllable.

Let $\tilde{\lambda}$ and λ be eigenvalues of K and A respectively, then

$$\tilde{\lambda}=\frac{1}{2}c_{00}+c_{10}\lambda+c_{20}\lambda^2$$

and let $x=Re\lambda$ and $y=Im\lambda$ and after simplification we get

$$Re\tilde{\lambda}=\frac{1}{2}(\gamma_{00}+\gamma_{10}x+\gamma_{20}x^2+\gamma_{01}y+\gamma_{20}y^2+\gamma_{11}xy)$$

Clearly $Re\tilde{\lambda}<0$, because $\lambda \in S$. Hence, the Lyapunov equation has a unique positive definite solution H.

Let $c_{11}>0$, equation (2.20) is equivalent to n^2 linear equations, whose $n^2\times n^2$ coefficient matrix has

$$\gamma_{00}+\gamma_{10}x+\gamma_{20}x^2+\gamma_{01}y+\gamma_{02}y^2+\gamma_{11}xy$$

$$x=Re\lambda_i,\ y=Im\lambda_j, \text{ and } i,j=1,2,\ldots,n$$

as its eigenvalues. Since $\lambda \in S$, (2.19) has a unique solution. From theorem 2.5, for any positive definite M, their exist a positive definite H solution of (2.19).

Let
$$M_t = tBB^* + (1-t)M$$
$$0 \le t \le 1$$

Clearly, M_t is positive definite for $0 \le t < 1$ and positive semidefinite for $t=1$. Equation (2.19) has unique solution H_t and the eigenvalues of H_t are real and vary continuously with t. Hence if we prove that H_t never becomes singular, we complete the prove. Let

$$H_t = \begin{pmatrix} \hat{H}_t & 0 \\ 0 & 0 \end{pmatrix}$$

we substitute $H=H_t$ in (2.19). Then every $n \times n$ element of the matrices on the left of (2.19) is zero except that of $c_{11}AHA^*$. Hence we arrive at the contradiction to the fact $-M_t$ is negative semidefinite, since $c_{11}AHA^*$ is positive semidefinite, because $c_{11} > 0$.

The solution to equation (2.20) will be called *the generalized controllability gramian.* The above theorem will be used to find Lyapunov-type matrix equations for sectors, circles, ellipses, parabolas and vertical strips in the following examples. In all examples, symmetric regions with respect to the x-axis are chosen because the poles of any system are symmetric with respect to the x-axis.

Example 2.1 Pole clustering inside a sector

Consider the sector shown in figure 2.1a, and described by

$$S = \{(x,y): -y + mx + b < 0\}$$

The condition $\lambda_{02} + \lambda_{20} \ge 0$ is satisfied because $\lambda_{02} + \lambda_{20} = 0 + 0 = 0$. Thus

$$c_{00}=b,\quad c_{10}=\bar{c}_{01}=\frac{1}{2}(m-i) \text{ and } c_{02}=c_{20}=c_{11}=0$$

and (2.19) becomes

$$2bH+(m-i)AH+(m+i)HA^{*}=-M \tag{2.21}$$

Example 2.2 Pole clustering inside a circle

Consider the circular region with radius r and center at $-\alpha$ shown in figure 2.1b and described by

$$S=\{(x,y):(x+\alpha)^2+y^2-r^2<0\}$$

The condition $\lambda_{02}+\lambda_{20}=1+1=2\geq 0$ is satisfied.

Thus $c_{00}=\alpha^2-r^2,\quad c_{10}=c_{01}=\alpha,\quad c_{11}=1,$ and $c_{20}=c_{02}=0$

and (2.19) becomes

$$\alpha(AH+HA^{*})+AHA^{*}+(\alpha^2-r^2)H=-M \tag{2.22}$$

Example 2.3 Pole clustering inside an ellipse

Consider the region inside the ellipse shown in figure 2.1c and described by

$$S=\left\{(x,y):\frac{(x+\alpha)^2}{a^2}+\frac{y^2}{b^2}-1<0\right\}$$

The condition $\gamma_{02}+\gamma_{20}=1+\left(\frac{a}{b}\right)^2\geq 0$ is satisfied.

Let $c=\left(\frac{a}{b}\right)^2$ then

$$c_{00}=\alpha^2-a^2,\quad c_{01}=c_{10}=\alpha,\quad c_{02}=c_{20}=\frac{1}{4}(1-c) \text{ and } c_{11}=\frac{1}{2}(1+c)$$

and (2.19) becomes

$$(\alpha^2 - a^2)H + \alpha(AH + HA^*) + \frac{1}{4}(1-c)[A^2H + H(A^*)^2] + \frac{1}{2}(1+c)AHA^* = -M \quad (2.23)$$

Example 2.4 Pole clustering inside a hyperbola

Consider the hyperbola shown in figure 2.1d, and described by

$$S = \left\{(x,y): \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 < 0\right\}$$

The condition $\gamma_{02} + \gamma_{20} \geq 0$ is not always satisfied

$$\gamma_{02} + \gamma_{20} = 1 - \left(\frac{a}{b}\right)^2 \geq 0 \Rightarrow -1 \leq \frac{a}{b} \leq 1$$

This means that the asymptotes of the hyperbola should have slopes between -1 and 1. Let $c = \left(\frac{a}{b}\right)^2$, then

$$c_{00} = -a^2, \quad c_{01} = c_{10} = 0, \quad c_{02} = c_{20} = \frac{1}{4}(1+c), \text{ and } c_{11} = \frac{1}{2}(1-c)$$

and (2.19) becomes

$$-a^2H + \frac{1}{4}(1+c)[A^2H + H(A^*)^2] + \frac{1}{2}(1-c)AHA^* = -M \quad (2.24)$$

Example 2.5 Pole clustering inside a parabola

Consider the region inside the parabola shown in figure 2.1e and described by

$$S = \{(x,y): y^2 + k(x+\alpha) < 0\}$$

The condition $\gamma_{02} + \gamma_{20} = 1 + 0 = 1$ is satisfied. Thus

$$c_{00} = \alpha k, \quad c_{01} = c_{10} = \frac{1}{2}k, \quad c_{02} = c_{20} = \frac{-1}{4}, \text{ and } c_{11} = \frac{1}{4}$$

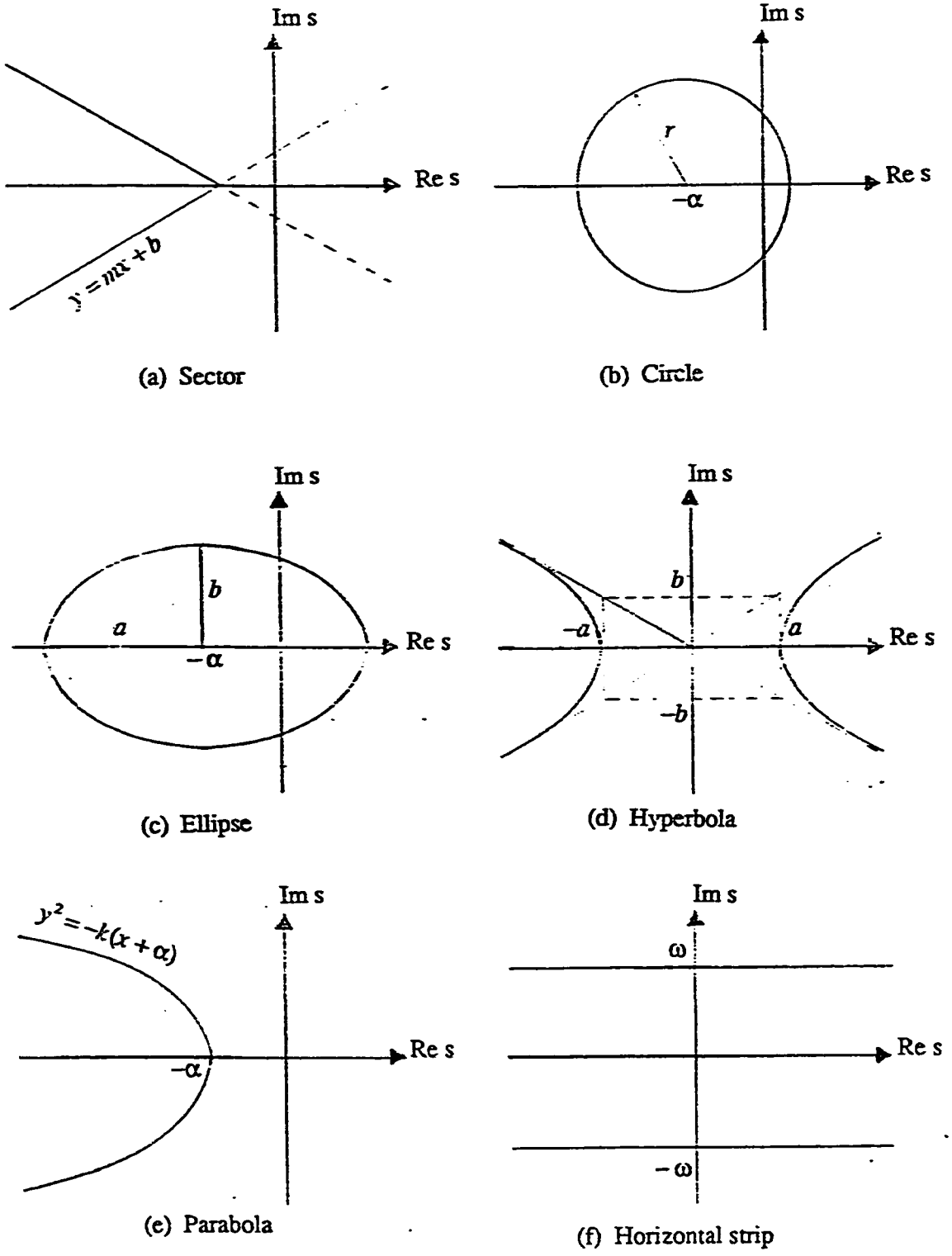

Fig. 2.2 : Different examples of region S.

and (2.19) becomes

$$\alpha kH + \frac{1}{2}k(AH + HA^*) - \frac{1}{4}[A^2H + H(A^*)^2] + \frac{1}{2}AHA^* = -M \tag{2.25}$$

Example 2.6 Pole clustering inside a horizontal strip

Consider the region inside the horizontal strip shown in figure 2.1f, and described by

$$S = \{(x,y): -w^2 + y^2 < 0\}$$

The condition $\lambda_{02} + \lambda_{20} = 1 + 0 = 1$ is satisfied. Thus

$$c_{00} = -w^2, \quad c_{01} = c_{10} = 0, \quad c_{02} = c_{20} = \frac{-1}{4}, \text{ and } c_{11} = \frac{1}{2}$$

and (2.19) becomes

$$-w^2H - \frac{1}{4}[A^2H + H(A^*)^2] + \frac{1}{2}AHA^* = -M \tag{2.26}$$

2.3 Generalized Balanced Model Reduction

Examining the balanced model reduction, we note the following. The poles of a stable continuous system are clustered in the left half plane and the system satisfies the standard continuous-time Lyapunov equation. By using the continuous-time Lyapunov equation for model reduction, the reduced order model will be stable, i.e. the poles of the reduced order model are clustered in the left half plane. Similarly the poles of stable discrete-time system are clustered inside the unit circle and the system satisfies the standard discrete-time Lyapunov equation. If the discrete-time Lyapunov equation is used for model reduction, the

reduced order model will be stable, i.e. the poles of the reduced order model will be clustered inside the unit circle. Hence, if the poles of the system are clustered in some region and if a Lyapunov type equation that is satisfied by the system can be found, then this Lyapunov equation may be used, instead of continuous-time or discrete-time Lyapunov equations, to produce a reduced order model with poles clustered in the same region.

We will restrict the regions to second order regions S which are described by equation (2.18). Second order regions are chosen because they are simple to describe and they represent the most regions of interest. The algorithm, for obtaining reduced order models with poles clustered in S, is similar to the balanced model reduction with replacing the controllability or observability gramian with the generalized controllability or observability gramian (2.20). The following algorithm gives parallel steps to the balanced model reduction algorithm given in [39].

Step 1 Solve for the generalized controllability gramian and the standard observability gramian

$$c_{00}P + c_{10}AP + c_{20}A^2P + c_{01}PA^* + c_{02}P(A^*)^2 + c_{11}APA^* = -BB^* \quad (2.27)$$

$$A^*Q + QA = -C^*C \quad (2.28)$$

Or solve for the standard controllability gramian and the generalized observability gramian

$$AP + PA^* = -BB^* \quad (2.29)$$

$$c_{00}Q + c_{10}A^*Q + c_{20}(A^*)^2Q + c_{01}QA + c_{02}QA^2 + c_{11}A^*PA = -C^*C \quad (2.30)$$

Step 2 Compute the Cholesky factors of the gramians

$$P = L_c L_c'; \quad Q = L_o L_o' \tag{2.31}$$

where L_c and L_o denotes the lower triangular Cholesky factors of P and Q respectively.

Step 3 Compute the singular value decomposition of the product of the Cholesky factors

$$L_o' L_c = U \Sigma V' \tag{2.32}$$

Step 4 The balanced transformation matrix T, its inverse and the balanced system are given by

$$T = L_c V \Sigma^{\frac{-1}{2}}; \quad T^{-1} = \Sigma^{\frac{-1}{2}} U' L_o' \tag{2.33}$$

$$A_b = T^{-1} A T; \quad B_b = T^{-1} B; \quad C_b = C T \tag{2.34}$$

Step 5 Partition the balanced system (A_b, B_b, C_b) as

$$A_b = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}; \quad B_b = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}; \quad C_b = (C_1 \quad C_2); \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \tag{2.35}$$

and the reduced order model is (A_{11}, B_1, C_1).

In this algorithm, one standard Lyapunov equation is used to guarantee the stability of the reduced model and one generalized Lyapunov equation is used to guarantee clustering of the poles of the reduced order model in S. In step 1 of the algorithm, we use either the generalized controllability gramian or observability gramian. There is no known optimal way to choose either one of them. Nevertheless, they will be compared by simulation. Also in some cases such as the reduction of unstable systems, the two generalized gramians should be used. Otherwise, P or Q will not be positive definite.

The generalized balanced model reduction will produce a generically controllable and observable reduced order model because the weakly controllable and weakly observable states are neglected. The reduced order model will also be generically asymptotically stable because if we partition the observability gramian (2.28) as in (2.35), we get

$$A_{11}^{*}\Sigma_1 + \Sigma_1 A_{11} = -C_1^{*}C_1 \tag{2.36}$$

Since the reduced order model is generically observable, it is generically asymptotically stable from (2.36). The stability can also be shown using the controllability gramian (2.29). To determine if the poles of the reduced order model are clustered in S, consider the generalized controllability gramian (2.27) Using the partitioning (2.35) and after simplifications, we get

$$\begin{aligned} &c_{00}\Sigma_1 + c_{10}A_{11}\Sigma_1 + c_{20}A_{11}^2\Sigma_1 + c_{01}\Sigma_1 A_{11}^{*} + c_{02}\Sigma_1 (A_{11}^{*})^2 + c_{11}A_{11}\Sigma_1 A_{11}^{*} \\ &\quad = -B_1 B_1^{*} - c_{20}A_{12}A_{21}\Sigma_1 - c_{02}\Sigma_1 A_{21}^{*}A_{12}^{*} - c_{11}A_{12}\Sigma_2 A_{12}^{*} \end{aligned} \tag{2.37}$$

The reduced order model has poles clustered in S, if the right hand side of (2.37) is negative definite. The terms $-B_1B_1^{*}$ and $-c_{11}A_{12}\Sigma_2 A_{12}^{*}$ are negative semidefinite since $c_{11} = \frac{1}{2}(\lambda_{02} + \lambda_{20}) \geq 0$. The other two terms, $-c_{20}A_{12}A_{21}\Sigma_1 - c_{02}\Sigma_1 A_{21}^{*}A_{12}^{*}$, are not known to be definite. But if $c_{02} = c_{20} = 0$, then (2.37) is satisfied and the reduced order model is proven to have poles clustered in S. The restriction $c_{02} = c_{20} = 0$ will produce first order regions or circles. Finally, we point out that the reduced order model is not internally balanced, this is clear from (2.37).

2.4 Examples

In the following, the generalized model reduction technique is applied to three examples. In the first one, a ninth order transfer function model is reduced to a third order model. The poles of the system are clustered around the line s=-1. Using balanced model reduction, the poles of the reduced order model are clustered around the line s=-0.3. The generalized model reduction is used to restrict the poles to be clustered in an ellipse with center at s=-1, and major axis with length 8 along y-axis and minor axis with length 0.2 along x-axis. In the second example, a sixth order system with three-inputs and three-outputs is used. The poles of the system are clustered in a sector with slope 0.2867. After reducing the order of the system to a third order using the standard balancing technique, two poles of the reduced order model became outside the sector. Then, the generalized balanced model reduction is applied to produce a reduced order model with poles clustered inside the sector. Example three is a reduction of sixth order model of fighter described in [47]. This model has two unstable poles. Therefore, the balanced model reduction can not be applied. However, the generalized balanced model reduction can be applied to reduce it. In this example, the generalized balancing is applied four times. In each time, the reduction is performed with respect to a different region.

Example 2.7

The transfer function of a closed-loop system is given by [42]:

$$G(s)=\frac{(s+4)(s+25)[(s+3)^2+8]}{(s+0.9846)[(s+0.9998)^2+15.9928][(s+1.0018)^2+9.032][(s+0.9936)^2+3.95][(s+1.0125)^2+1.0255]}$$

It is obvious that all the poles have almost the same real part. When the balanced model

reduction is used, the reduced order model is

$$G_{bv}(s)=\frac{-0.2466(s-2.2591)}{(s+0.2948)^2+0.7100}$$

and its poles are located at $-0.2948\pm0.8426i$ as seen in figure 2.2a . Clearly, the poles of the reduced order model are far a way from the poles of the full order model. In order to get a reduced order model with poles located in the same region as the full order model, we choose an ellipse that contains all the poles. This ellipse has center at $s=-1$, major axis has length 8 along the y-axis and the minor axis has length 0.2 along the x-axis as is shown in figure 2.2a . The generalized balanced model reduction is applied twice. First the generalized controllability gramiam and the observability gramian are used. The reduced order model is

$$G_{gbv1}(s)=\frac{-0.3996(s-2.2304)}{(s+0.9758)^2+0.8789}$$

When the controllability gramiam and generalized observability gramiam are used, the reduced order model becomes

$$G_{gbv2}(s)=\frac{-0.4044(s-2.2417)}{(s+0.9992)^2+0.8315}$$

Clearly, from figure 2.2a both reduced order models have their poles clustered inside the ellipse.

Figure 2.2b shows a plot of the maximum singular value of the transfer function of the reduction error for each method versus frequency. The standard balanced model reduction has a smaller error than the generalized one. However, this is expected, because we are restricting the left half plane to an ellipse. Hence, the price that we pay for pole restriction, is this increase in the error frequency response. In [60], a similar statement was made.

That is, when the region that encloses the poles is increased, the error response is decreased and vice verse. Therefore, as rule of thumb, it is better to make the region that encloses the poles as big as possible in order to make the error decreases.

Figure 2.2c shows the step responses of the full order, the balanced and generalized balanced reduced order models. The step response of the balanced reduced model is closer to the step response of the full order model, but there is a big difference in the transient due to the oscillations in the balanced reduced order model. While there is a big difference in the steady state of the generalized balanced reduced model and the full order model, they have similar shapes and transients.

Example 2.8

Consider the following sixth order system with three-inputs and three-outputs given in [42]:

$$A = \begin{pmatrix} -0.3110 & 1.0280 & 0 & 0 & 2.0000 & -6.3000 \\ -0.1100 & -1.2560 & 0 & -6.3000 & -1.0370 & 4.1000 \\ 0 & 0 & -1.3000 & 0 & 0.0048 & 0 \\ 0 & 0.0025 & 0 & -0.1760 & -0.0030 & 0 \\ 0 & -0.0380 & -6.4000 & -3.7500 & -0.97500 & 0 \\ 0.0250 & -0.0400 & 0 & 0 & -0.0091 & -1.8400 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0300 & 0 & 0 \\ 0 & 0 & 0.0520 \\ 0.0039 & 0 & 0 \\ 0 & 0.0025 & 0 \\ 0.0244 & 0 & 0 \\ 0 & 0 & 0.0370 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 1.0000 & 0 & 2.2400 & 0 \\ 0 & 2.2400 & 0 & 1.0000 & 0 & 0 \\ 2.2400 & 0 & 0 & 0 & 0 & 1.0000 \end{pmatrix}$$

The poles of the system are :

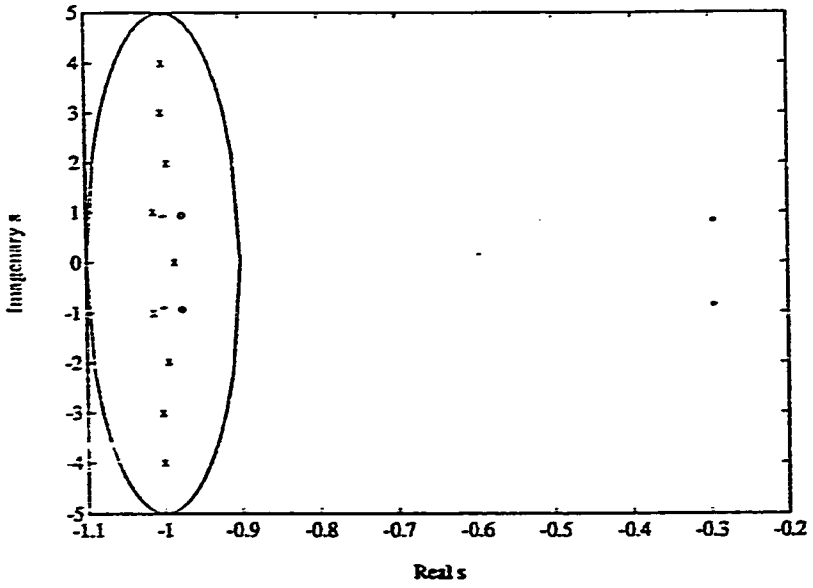

Fig. 2.2a : Poles of $G(s)$, $G_{br}(s)$, $G_{gbr1}(s)$ and $G_{gbr2}(s)$: (x) poles of $G(s)$, (*) poles of $G_{br}(s)$, (o) poles of $G_{gbr1}(s)$, (+) poles of $G_{gbr2}(s)$.

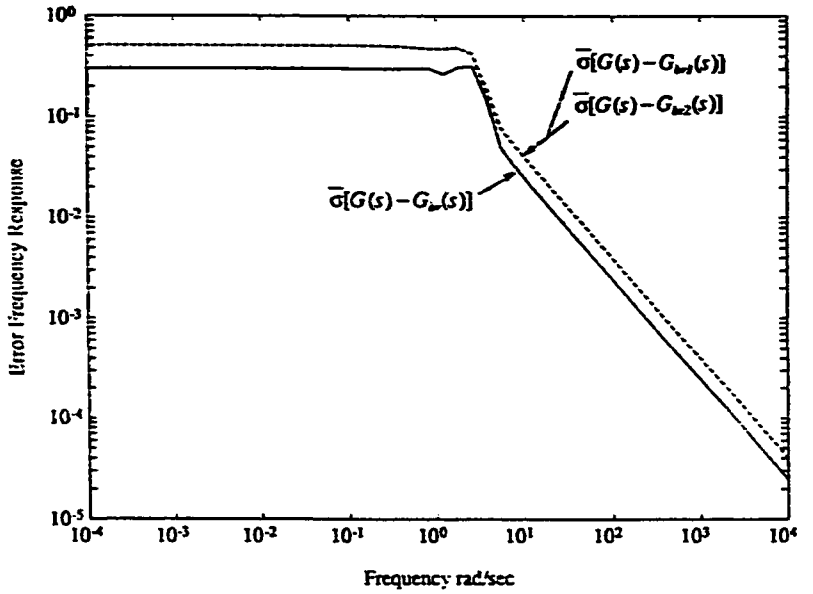

Fig. 2.2b : Error frequency response.

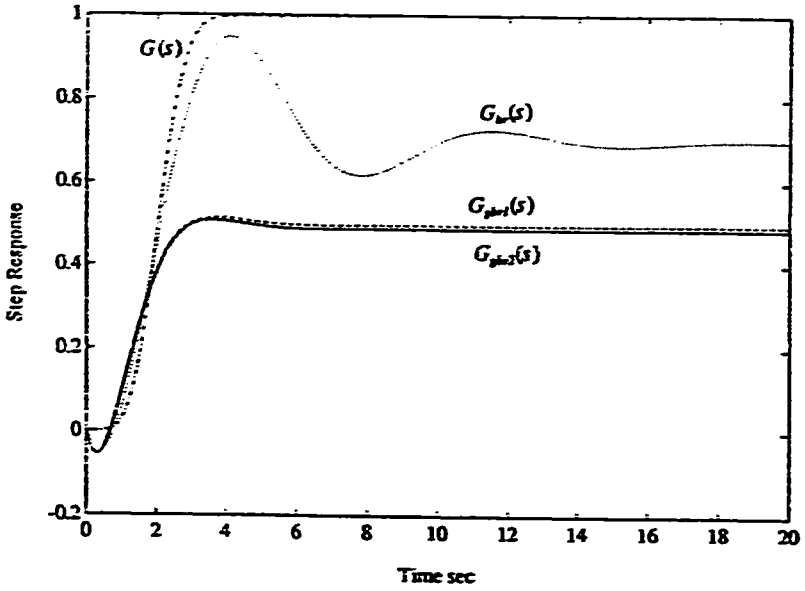

Fig. 2.2c : Step response of $G(s)$, $G_{lv}(s)$, $G_{glv1}(s)$ and $G_{glv2}(s)$.

$-0.1621, -1.4942 \pm 0.4188i, -0.4761, -1.1985, -1.0328,$

They are clustered inside a sector with slopes ± 0.2789 as shown in figure 2.3a. The reduced order model using balancing is

$$A_{br} = \begin{pmatrix} -0.0807 & 0.1678 & 0.1828 \\ -0.0484 & -0.5654 & -0.9826 \\ -0.0381 & 1.0616 & -2.1617 \end{pmatrix}$$

$$B_{br} = \begin{pmatrix} 0.1094 & -0.2385 & -0.0676 \\ -0.1316 & -0.1482 & 0.3705 \\ 0.0612 & 0.0688 & -0.3271 \end{pmatrix}$$

$$C_{br} = \begin{pmatrix} 0.0525 & -0.0181 & 0.0011 \\ -0.0122 & 0.3446 & 0.1506 \\ 0.2655 & -0.2398 & -0.3046 \end{pmatrix}$$

The poles of the reduced order model are $-0.0921, -1.3578 \pm 0.6375i$. They are not inside the sector, as can be seen in figure 2.3a. The generalized balanced model reduction will be used to produce a reduced order model with poles clustered inside this sector. But the generalized Lyapunov equation for a sector (2.21) gives a solution with complex coefficients. The balancing transformation T will also be complex. Hence, the transfer function of the reduced order model will be complex. This is not acceptable since there is no real system with complex coefficients. Thus another approach will be used.

Consider the hyperbola (2.24) with a tending to zero and $c = (0.2789)^2$. This is the degenerate case of hyperbola. It is two intersecting straight lines of slopes ± 0.2789 and passing through the origin. Now, we can use the generalized Lyapunov equation of the hyperbola to make the poles of the reduced order model cluster inside the sector. Since, there are two sectors one in the left half plane and the other in the right half plane, there is a possibility of getting poles of the reduced order model in the right half plane. However, This will not happen, because the other Lyapunov equation (2.28) or (2.29) will ensure

that the poles of the reduced order model are clustered in the left half plane. Applying this method, the reduced order model using equations (2.27) and (2.28) is

$$A_{gbr1} = \begin{pmatrix} -0.7314 & -0.8704 & -0.6702 \\ -0.4729 & -0.8979 & -0.9020 \\ 0.0893 & 0.0878 & -1.3568 \end{pmatrix}$$

$$B_{gbr1} = \begin{pmatrix} 0.0370 & -0.0376 & -0.0378 \\ -0.0029 & 0.0673 & -0.0351 \\ -0.0344 & -0.0623 & 0.1048 \end{pmatrix}$$

$$C_{gbr1} = \begin{pmatrix} 0.0566 & -0.1977 & 0.1064 \\ -1.2744 & -1.8762 & -0.6177 \\ 2.7432 & 1.7545 & 1.2146 \end{pmatrix}$$

and the reduced order model using equations (2.29) and (2.30) is

$$A_{gbr2} = \begin{pmatrix} -0.2688 & 0.3011 & -0.0969 \\ 0.3999 & -1.3684 & 0.2084 \\ 0.0541 & -0.6671 & -1.3591 \end{pmatrix}$$

$$B_{gbr2} = \begin{pmatrix} -0.1281 & 1.2766 & 1.1042 \\ 0.6795 & 0.3179 & -3.4828 \\ 2.1417 & 0.4086 & -0.3267 \end{pmatrix}$$

$$C_{gbr2} = \begin{pmatrix} -0.0136 & -0.0010 & 0.0197 \\ 0.0171 & -0.0601 & 0.0316 \\ -0.0672 & 0.0085 & 0.0533 \end{pmatrix}$$

Figure 2.3a shows that the poles of the two generalized balanced reduced order models are clustered inside the sector. Figure 2.3b shows the error frequency response for the two generalized balanced methods and balancing.

Example 2.9

A sixth order model of a fighter is used in this example. A full description of this model is given in [47]. The model is

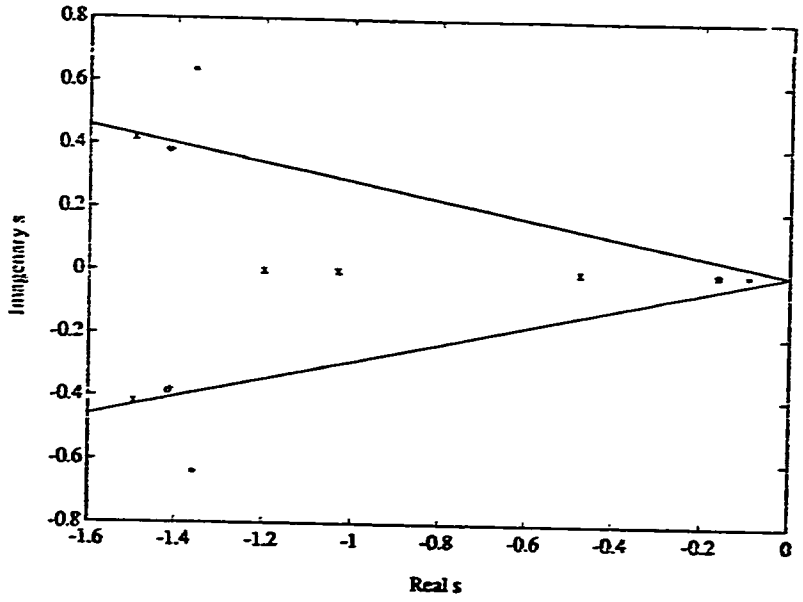

Fig. 2.3a : Poles of the full order model and the reduced order models : (x) poles of A, (*) poles of A_{br}, (o) poles of A_{gbr1}, (+) poles of A_{gbr2}.

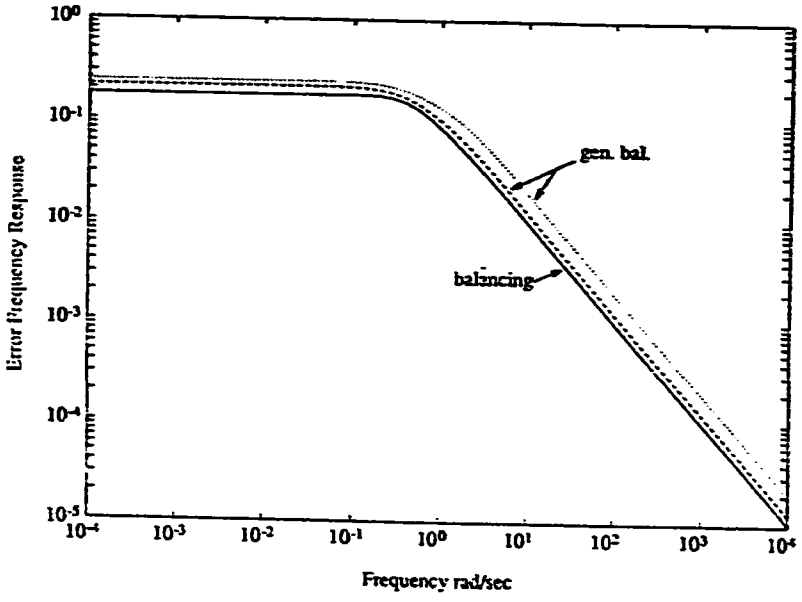

Fig. 2.3b : Error frequency response.

$$A=\begin{pmatrix} -0.0226 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.0050 \\ 0.0123 & 11.7200 & -2.6316 & 0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.0000 \end{pmatrix}$$

$$B'=\begin{pmatrix} 0 & 0 & 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30 \end{pmatrix}, \quad C=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and its poles are $-5.6757, 0.6898 \pm 0.2488i, -0.2578, -30.0, -30.0$. This model has two unstable poles. Thus the controllability and observability gramians do not exist and the solutions of the standard Lyapunov equations are indefinite. Thus, they can not be used in balanced model reduction. To produce a positive definite P and Q, we should use the generalized controllability and the generalized observability gramians (2.27 and 2.30) should be used with region S that includes all the poles of the model. In this example, four regions are chosen to examine the effect of S on model reduction. The first region denoted by S_1 is the region inside the horizontal strip intersecting the imaginary axis at $s=\pm 0.5$ shown in figure 2.4a. The second region S_2 is the region inside the ellipse whose center is at $s=-15$ and a major axis of length 40 along x-axis and a minor axis of length 1 along y-axis shown in figure 2.4b. The third region S_3 is the region inside the parabola $y^2=-0.018(x-5)$ shown in figure 2.4c and the fourth region S_4 is the region inside the parabola $y^2=\frac{(0.3)^2}{35}(x+35)$ as shown in figure 2.4d. The reduced order model for each region are given respectively as:

$$A_{S_1} = \begin{pmatrix} 0.9635 & 0.3534 & 0.0205 & 0.0373 \\ -0.3914 & 0.3729 & -0.1017 & 0.1514 \\ -0.0375 & -0.1158 & -5.6735 & -0.0230 \\ -0.0227 & 0.1540 & -0.0766 & -5.9818 \end{pmatrix}$$

$$B_{S_1} = \begin{pmatrix} 3.0447 & -2.1172 \\ 2.2828 & -1.5712 \\ 2.2617 & -1.6333 \\ -0.1691 & 0.0526 \end{pmatrix}$$

$$C_{S_1} = \begin{pmatrix} -1.4341 & 0.1458 & 2.3568 & -0.1114 \\ -3.2497 & 3.0096 & 1.5538 & -1.2736 \end{pmatrix}$$

$$A_{S_2} = \begin{pmatrix} 0.7817 & 0.2633 & -0.0037 & -0.0354 \\ -0.2681 & 0.5959 & 0.0413 & -0.0498 \\ 0.0042 & 0.0447 & -0.2591 & -0.0251 \\ -0.0011 & -0.0011 & 0.0136 & -30.0000 \end{pmatrix}$$

$$B_{S_2} = \begin{pmatrix} -3.8061 & 2.6449 \\ -3.1042 & 2.1468 \\ 1.1672 & -0.7788 \\ -1.1112 & 0.8604 \end{pmatrix}$$

$$C_{S_2} = \begin{pmatrix} 1.5368 & -0.5806 & -0.3559 & 0.9678 \\ 4.2752 & -3.8495 & 1.4443 & 1.0186 \end{pmatrix}$$

$$A_{S_3} = \begin{pmatrix} 0.7738 & 0.2598 & -0.0099 & -0.0022 \\ -0.2660 & 0.6041 & -0.0153 & -0.0387 \\ 0.0012 & 0.0008 & -30.0000 & 0.0161 \\ -0.0040 & -0.0406 & -0.0020 & -0.2562 \end{pmatrix}$$

$$B_{S_3} = \begin{pmatrix} 3.8775 & -2.6898 \\ 3.1361 & -2.1647 \\ -1.1072 & 0.8631 \\ 1.2365 & -0.8291 \end{pmatrix}$$

$$C_{S_3} = \begin{pmatrix} -1.5420 & 0.6123 & 0.9692 & -0.3059 \\ -4.3137 & 3.9290 & 1.0182 & 1.5032 \end{pmatrix}$$

$$A_{S_4}=\begin{pmatrix} 0.7392 & 0.2524 & -0.0035 & -0.0178 \\ -0.2550 & 0.6398 & -0.0209 & -0.0199 \\ -0.0032 & -0.0218 & -0.2647 & 0.2084 \\ -0.0048 & -0.0064 & 0.1925 & -5.6682 \end{pmatrix}$$

$$B_{S_4}=\begin{pmatrix} -4.0971 & 2.8498 \\ -3.4009 & 2.3564 \\ -1.2388 & 0.8330 \\ -2.3599 & 1.7022 \end{pmatrix}$$

$$C_{S_4}=\begin{pmatrix} 1.5875 & -0.7134 & 0.3604 & -2.3595 \\ 4.6067 & -4.1786 & -1.5485 & -1.6629 \end{pmatrix}$$

The error frequency response for each method is shown in figure 2.4e. It is clear for the graph, that the error frequency response depends highly on the chosen region. Region S_3 gives the minimum error over all the other three regions for this example. The poles of the reduced order models for every region are clustered inside the same region as shown in figures 2.4a, b, c and d.

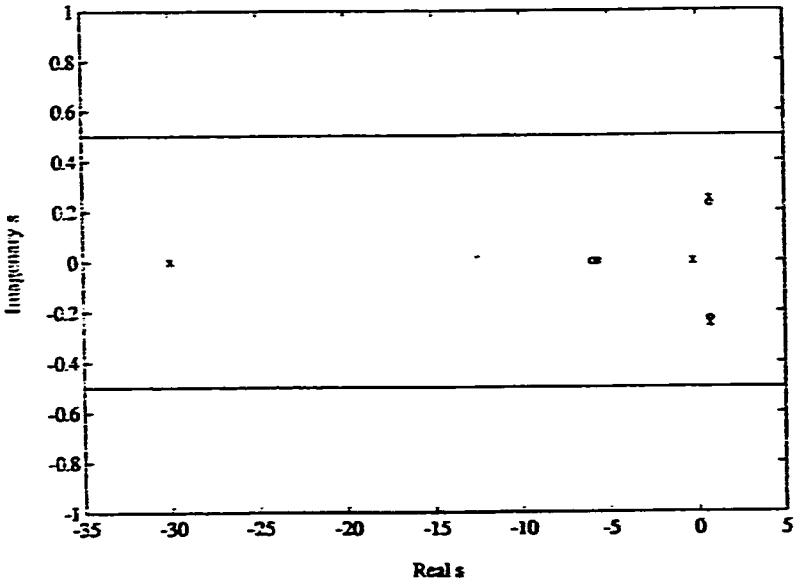

Fig. 2.4a : The poles of the full and the reduced order systems are clustered inside a horizental strip. (x) poles of the full order model, (o) poles of the reduced order model.

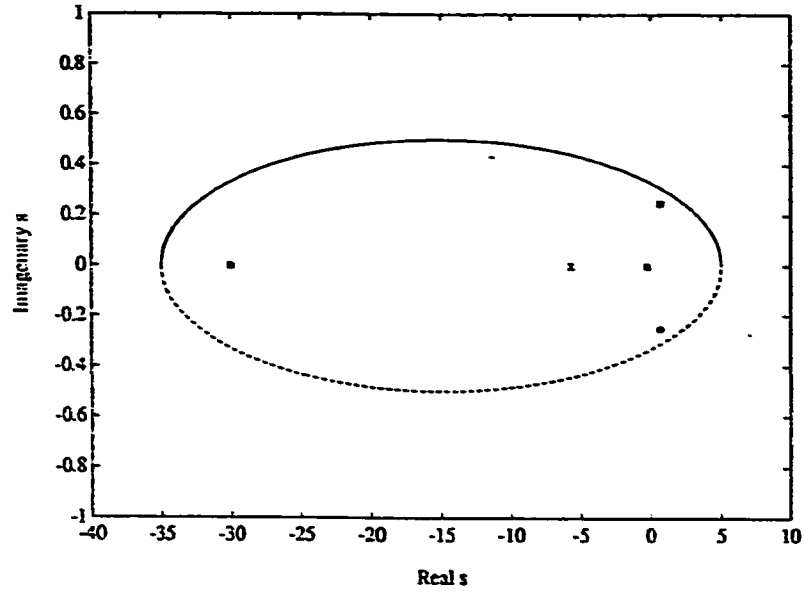

Fig. 2.4b : The poles of the full and the reduced order systems are clustered inside an ellipse. (x) poles of the full order model, (o) poles of the reduced order model.

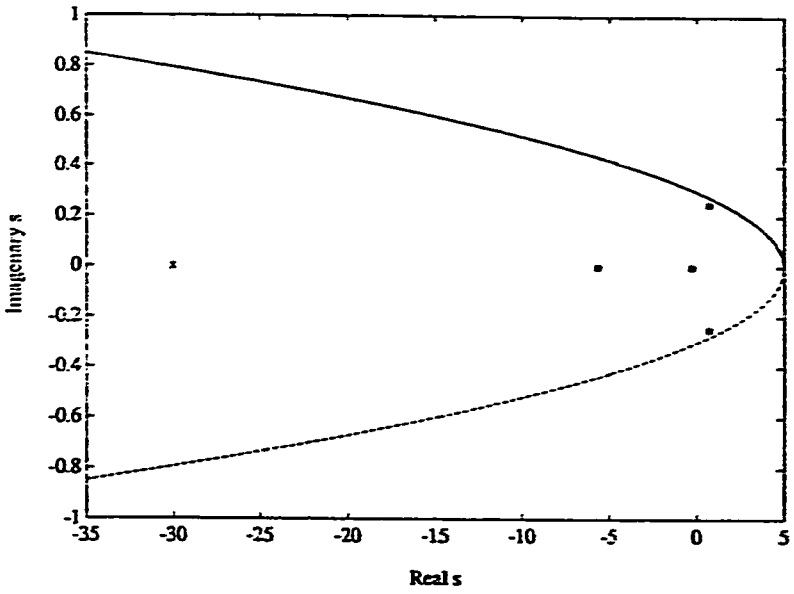

Fig. 2.4c : The poles of the full and the reduced order systems are clustered inside a parabola. (x) poles of the full order model, (o) poles of the reduced order model.

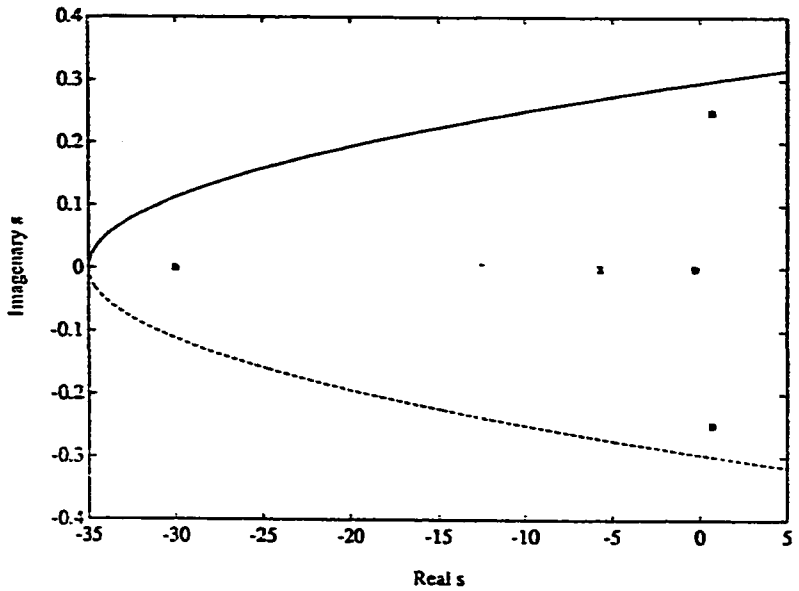

Fig. 2.4d : The poles of the full and the reduced order systems are clustered inside a parabola. (x) poles of the full order model, (o) poles of the reduced order model.

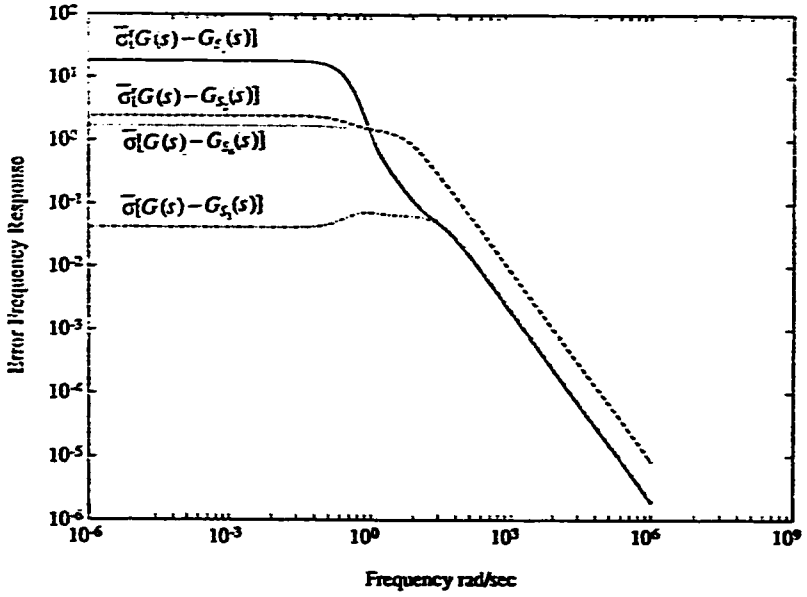

Fig. 2.4e : Error frequency response.

CHAPTER 3

MODEL REDUCTION USING BALANCED REALIZATION WITH IMPROVED FREQUENCY BEHAVIOR

3.1 Introduction

In model reduction, sometimes it is important that reduction error is small at a certain frequency band. This is especially important when using the reduced order model in feedback control system design. There an accurate approximation of the full order system is needed at the crossover frequency region. This idea motivated some authors to use frequency weighting for system approximation. Moreover, the balanced realization techniques give good approximation of the impulse response but have a drawback of giving a large steady state error for step input. This indicates that the approximation is better for high frequencies than low frequencies [4]. For control system design, however, the low frequencies behavior is more important than high frequency one and hence a good approximation at low frequencies is highly important.

In this chapter, a simple modification to the reduction technique using balanced realization is proposed to improve its frequency behavior. It is based on the observation that the error frequency response is similar to the frequency response of a low pass filter. Therefore, a frequency transformation is used to transform it to high pass, band stop or band reject filter. Frequency transformation techniques are widely used in filter design and

can be found in any filter design textbook. However, not all frequency transformations can be used for purpose of modifying the frequency behavior of the reduction error. A restriction on the transformation to be used is that it is one to one. This means that the frequency transformation formulas for band reject and band stop can not be used. This is because these transformations are second order and the inverse does not exist or it is not unique. Therefore, we will concentrate only on the transformation from high pass to low pass or low pass to high pass.

The chapter is organized as follows: In section 3.2, bilinear transformation and its properties will be reviewed. In section 3.3, the proposed technique and some of its properties are presented. Finally, in section 3.4, the proposed technique is illustrated by numerical examples.

3.2 Bilinear Transformation

In this section, a review the bilinear transformation and its properties will be given. Also, we will prove that the controllability gramian of a system is the same as the generalized controllability gramian (2.22) of the transformed system with respect a circle determined from the bilinear transformation.

A bilinear transformation is defined as a mapping from the complex plane to the complex plane and is of the form

$$s = \frac{az + b}{cz + d} \tag{3.9}$$

with $w = |ad - bc| > 0$

The following lemma is sometimes useful in constructing bilinear transformation between given domains and in the determination of the region of pole clustering of the transformed system. First, we observe that a circle or a straight line in the plane separates the plane into two domains, D^* and D^{**} both having a circle or a straight line as the boundary. These domains are called complementary domains of the circle or the straight line and are illustrated in figure 3.1.

Lemma 3.1 [62]

A bilinear transformation always maps a circle to a circle or a straight line and always maps a straight line to a circle or a straight line.

Further, let k be a circle or a straight line in the s-plane having complementary domains D and D'. Let $s = \frac{\alpha z+b}{\alpha+d}$ map k to k^* in the z-plane. Then $s = \frac{\alpha z+b}{\alpha+d}$ maps D onto one of the complementary domains of k^*, and it maps D' onto the other complementary domain of k^*.

The following two lemmas give $G(s)$ after bilinear transformation and show that the bilinear transformation preserves linearity, system order and minimality.

Lemma 3.2 [60]

Let $G(s)$ be a stable transfer function matrix of McMillan degree n. Let $s=h(z)$ be a bilinear transformation that maps the left half plane onto D

$$h(z) = \frac{az+d}{cz+d}$$

with $w = |ad - bc|^{1/2} > 0$ and let (A,B,C,D) be a minimal realization of $G(s)$. Then a realization of $G(h(z))$ (or simply $G(z)$) of the same McMillan degree is given by (A_t, B_t, C_t, D_t) where

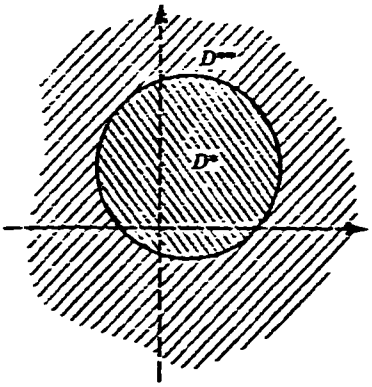

Fig. 3.1a Comlementary domaims determined by a circle.

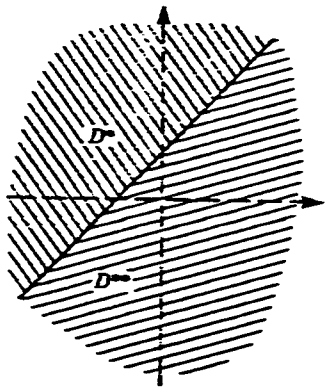

Fig. 3.1b Complementary domains determined by a stright line.

$$A_t = (aI - cA)^{-1}(dA - bI) \tag{3.10}$$

$$B_t = w(aI - cA)^{-1}B \tag{3.11}$$

$$C_t = wC(aI - cA)^{-1} \tag{3.12}$$

$$D_t = D + cC(aI + cA)^{-1}B \tag{3.13}$$

Lemma 3.3 [60]

If $w = |ad - bc|^{1/2} > 0$, then (A,B,C,D) is minimal if and only if (A_t,B_t,C_t,D_t) is minimal.

A bilinear transformation maps a straight line into a circle or a straight line which can be considered as a circle in a wide sense, i.e a circle with radius infinity. Now, we want to find the center and the radius of the circle that is produced from transforming the imaginary axis using the bilinear transformation $s = \frac{az+b}{cz+d}$. This can be easily found by calculating z at $s=0$ and $s=\infty$. These two points are the intersection of a diameter with the circle and they are given as follows

$$z_0 = -\frac{b}{a} \text{ at } s = 0 \tag{3.14}$$

$$z_\infty = -\frac{d}{c} \text{ at } s = \infty \tag{3.15}$$

Let r and x_0 be the radius and the center of the circle respectively. Hence,

$$r = \frac{1}{2}|z_\infty - z_0| = \left|\frac{ad - bc}{2ac}\right| \tag{3.16}$$

and

$$x_0 = \frac{1}{2}(z_\infty + z_0) = -\frac{ad + bc}{2ac} \tag{3.17}$$

Note that x_0 is a complex number. If a,b,c and d are real numbers, then x_0 is a real number. This means that the circle is symmetric with respect to the x-axis.

Now, we are ready to prove that the controllability gramian of the system (A,B,C,D) is

the same as the generalized controllability gramian of the system (A_t,B_t,C_t,D_t) with S being a circle of radius r and center x_0. This is given in the following theorem.

Theorem 3.1

The following two Lyapunov equations have the same solution P.

$$AP+PA'+BB'=0 \tag{3.18}$$

$$-x_0(A_tP+PA'_t)+A_tPA'_t+(x_0^2-r^2)P+\frac{w^2}{2ac}B_tB'_t=0 \tag{3.19}$$

where A_t, B_t are defined in lemma 3.2 with a,b,c and d real and r and x_0 are defined in (3.16) and (3.17) respectively.

Proof

Using the inverse transformation, we get A and B in terms of A_t and B_t as follows

$$A=(cA_t+dI)^{-1}(aA_t+bI) \tag{3.20}$$

$$B=w(cA_t+dI)^{-1}B_t \tag{3.21}$$

substituting (3.20) and (3.21) in (3.18) gives

$$(cA_t+dI)^{-1}(aA_t+bI)P+P(aA_t+bI)(cA_t+dI)^{-1}$$

$$+w^2(cA_t+dI)^{-1}B_tB'_t(cA_t+dI)^{-1}=0 \tag{3.22}$$

Multiplying (3.22) by (cA_t+dI) from the right and by $(cA_t+dI)'$ from the left, gives

$$(aA_t+bI)P(cA'_t+dI)+(cA_t+dI)P(aA'_t+bI)+w^2B_tB'_t=0$$

After expanding and simplification, we get

$$A_tPA'_t+\frac{ad+bc}{2ac}(A_tP+PA'_t)+\frac{bd}{ac}P+\frac{w^2}{2ac}B_tB'_t=0 \tag{3.23}$$

From (3.16) and (3.17), we note that

$$x_0=-\frac{ad+bc}{2ac} \text{ and } x_0^2-r^2=\frac{bd}{ac}$$

After substituting in (3.23), we get (3.19).

This theorem has been proved for some special cases, Glover [23] for $a=b=d=1$, $c=-1$ and Al-Saggaf [4] for $a=\alpha$, $b=\alpha\beta$, $c=-1$ $d=\beta$.

Using duality, we can also prove that the observability gramian

$$A'Q+QA+C'C=0$$

and the generalized observability gramian of the transformed system with respect a circle

$$-x_0(A_t'Q+QA_t)+A_t'QA_t+(x_0^2-r^2)Q+\frac{w^2}{2ac}C_t'C_t=0$$

have the same solution Q. Moreover, similar result exists if the two regions are circles, i.e. if D_1 is transformed into D_2 using bilinear transformation then the generalized controllability (observability) gramian of the system with respect to D_1 is equal to a constant multiplied by the generalized controllability (observability) gramian of the transformed system with respect to D_2, as in the following corollary.

Corollary 3.1

Let $h(s)$ be bilinear transformation that maps D_1 into D_2. Consider the following two Lyapunov equations

$$-x_1(AP_1+P_1A')+AP_1A'+(x_1^2-r_1^2)P_1+BB'=0 \qquad (3.24)$$

$$-x_2(A_tP_2+P_2A_t')+A_tP_2A_t'+(x_2^2-r_2^2)P_2+BB_t'=0 \qquad (3.25)$$

where x_1 and r_1 are the center and the radius of D_1, x_2 and r_2 are the center and the radius of D_2 and A_t and B_t are defined in lemma 3.2 with a,b,c and d are real. Then

$$P_1=\gamma P_2 \qquad (3.26)$$

where γ is a real number.

Proof

Let $h(s)$ be decomposed into a composition of two bilinear transformations $h_1(s)$ and $h_2(s)$

$$h(s) = h_1(h_2(s))$$

where $h_1(s) = \frac{a_1 s + b_1}{c_1 s + d_1}$ is the bilinear transformation that maps D_1 into the left half plane and $h_2(s) = \frac{a_2 s + b_2}{c_2 s + d_2}$ is the bilinear transformation that maps the left half plane into D_2. Let the state space realization of a system be (A,B,C,D), the state space realization be (A_1,B_1,C_1,D_1) after applying $h_1(s)$ and the state space realization be (A_2,B_2,C_2,D_2) after applying $h_2(s)$.

Multiply equation (3.24) by $\frac{w_1^2}{2a_1c_1}$ where $w_1^2 = |a_1d_1 - b_1c_1|$ we get

$$-x_1\left[A_1\left(\frac{w_1^2}{2a_1c_1}P_1\right)+\left(\frac{w_1^2}{2a_1c_1}P_1\right)A'_1\right]+A_1\left(\frac{w_1^2}{2a_1c_1}P_1\right)A'_1+(x_1^2-r_1^2)\left(\frac{w_1^2}{2a_1c_1}P_1\right)+\frac{w_1^2}{2a_1c_1}B_1B'_1=0$$

and the controllability gramian of (A,B,C,D) is

$$AP + PA' + BB' = 0$$

from theorem 3.1 we can see

$$P = \frac{w_1^2}{2a_1c_1}P_1 \tag{3.27}$$

Similarly, multiply (3.25) by $\frac{w_2^2}{2a_2c_2}$ where $w_2^2 = |a_2d_2 - b_2c_2|$ we get

$$-x_2\left[A_2\left(\frac{w_2^2}{2a_2c_2}P_2\right)+\left(\frac{w_2^2}{2a_2c_2}P_2\right)A'_2\right]+A_2\left(\frac{w_2^2}{2a_2c_2}P_2\right)A'_2+(x_2^2-r_2^2)\left(\frac{w_2^2}{2a_2c_2}P_2\right)+\frac{w_2^2}{2a_2c_2}B_2B'_2=0$$

also form theorem 3.1, we see

$$P = \frac{w_2^2}{2a_2c_2} P_2 \tag{3.28}$$

Form (3.27) and (3.28) we get

$$P_1 = \gamma P_2 \quad \text{where} \quad \gamma = \frac{a_1 c_1 w_2^2}{a_2 c_2 w_1^2}$$

3.3 Balanced Model Reduction With Improved Frequency Behavior Using Bilinear Transformation

In control systems, the low frequency region is much important than the high frequency region. Unfortunately in model reduction techniques the error frequency response is large at low frequencies and becomes less and less as the frequency increases. This means that the reduced order model is not a good approximation to the system in the low frequency region and it becomes a good approximation as the frequency increases. Our objective in this section is to produce a reduced order model that is a good approximation in a desired frequency region.

The error frequency response is the same as the frequency response of a low pass filter. Hence, the error frequency response can be changed to some desired responses using the standard methods of transforming a low pass filter to high pass, band pass or band reject filters. These methods are known in filter design fields as *frequency transformation*. A review of frequency transformation formulae is given in table 3.1 [63].

As seen in the table, the transformations to band pass or band reject are not invertable. These transformations can not be used in the proposed methods because it their inverse is needed. The transformations to high or low pass filters can be written as

Table 3.1 Frequency Transformation [63].

Lowpass filter Transformation to	Substitute s_{norm} into the normalized lowpass filter transfer function
Lowpass (normalization)	$s_{norm} = \frac{s}{\omega_x}$
Highpas	$s_{norm} = \frac{\omega_x}{s}$
Bandpass	$s_{norm} = \frac{\omega_x}{BW}\left(\frac{s}{\omega_x} + \frac{\omega_x}{s}\right)$ $BW = \omega_{c2} - \omega_{c1}$ $\omega_{c2} > \omega_{c1}$
Bandreject	$s_{norm} = \frac{BW}{\omega_x\left(\frac{s}{\omega_x} + \frac{\omega_x}{s}\right)}$ $BW = \omega_{s2} - \omega_{s1}$ $\omega_{s2} > \omega_{s1}$
s_{norm} = Normalized Frequency s = Variable Frequency ω_x = Normalizing Frequency	

$$s = \frac{az+b}{cz+d} \quad \Leftrightarrow \quad z = \frac{-ds+b}{cs-a}$$

By choosing appropriate values of a,b,c and d the transformation will be to the low or high pass filters.

The method is basically is to transform the system to another domain say z-domain. Then, the model reduction is preformed in the z-domain. After that, the inverse transformation is used to return back the model to s-domain. This inverse transformation will act as a frequency transformation for the error frequency response. Therefore, a special care should be devoted in the design stage for this inverse transformation. If the parameters of the inverse transformation are chosen similar to the frequency transformation from low pass to high pass, then error response will be low at low frequencies and high at high frequencies. Hence, their will be a high matching between the reduced order model and the original model at the low frequency region which is the nominal working region for control system design.

The transformation, that will be used, is the bilinear transformation. As seen in the last section, the bilinear transformation has many nice properties. It transforms the poles of a stable system from left half plane into the inside or the outside of a circle, in the wide sense, of radius r and center x_0 given by (3.16) and (3.17) respectively. Let this region be called S. S can also be defined as the region at which the left half plane is mapped to by using the bilinear transformation. Bilinear transformation can also be used to transform low pass filters into low or high pass filters.

Assume, we have a transfer function $G(s)$ and we want to apply the proposed method on it. First, transform $G(s)$ into $G(z)$ using bilinear transformation. Then, apply the model reduction techniques on $G(z)$ to get a reduced order model in the z-domain $G_r(z)$. Generally, balanced model reduction is used to produce $G_r(z)$. Unfortunately balanced model

reduction does not guarantee pole clustering in S. Since, the inverse transformation will be used to get $G_r(s)$ by transforming $G_r(z)$ from the z-domain to the s-domain and the inverse transformation maps S into the left half plane. Therefore, if $G_r(z)$ has some poles outside S, they will be mapped to right half plane, hence the reduced order model $G_r(s)$ will not be stable. To solve this problem, generalized balanced model reduction should be used to produce the reduced order model in the z-domain. In the generalized balanced model reduction, the reduced order model can be restricted to have poles clustered in S. The only difference between balanced model reduction and the generalized one is that the generalized controllability (or observability) gramian (2.22) will be used instead of the normal controllability (or observability) gramian (2.4 or 2.5). The generalized controllability (or observability) gramain for S, which is a circle in the wide sense, needs x_0 and r given by (3.16) and (3.17) respectively. However, x_0 and r may not be finite numbers in the case of a straight line. This will lead to numerical problems. This problem is solved using theorem 3.1 . Since, the solution of the generalized controllability (or observability) gramian of the transformed systems is the same as the solution of the controllability (or observability) gramian of the original system. Hence, to get a reduced order model with poles clustered in S, use the controllability (or observability) gramian of the original model and the observbility (or controllability) gramian of the transformed model. The algorithm of this reduction method is as follows:

Step 1 Choose carefully the parameters of the bilinear transformation so that its inverse shapes the error as required.

Step 2 Find the transformed systems by calculating A_t, B_t, C_t, D_t in lemma 3.2.

Step 3 Perform the generalized model reduction that is described in chapter 2, with P and Q given by:

a) for continuous-time systems

$$AP + PA' + BB' = 0$$

$$QA + A'Q + C'C = 0$$

b) for discrete-time systems

$$APA' - P + BB' = 0$$

$$A'QA - Q + C'C = 0$$

Step 4 Use the inverse bilinear transformation and lemma 3.2 to get the reduced order model.

The above algorithm produces generically a stable reduced order model. To see this, let $h(s)$ be a bilinear transformation that maps the left half plane into D. By lemma 3.2 the poles of $G_i(s)=D_i+C_i(sI-A_i)^{-1}B_i$ are clustered in D. Using the results of chapter 2, the poles of the reduced order model of $G_i(s)$ are also clustered in D. Using lemma 3.2 again for the transformation of the reduced order model to the s-plane, the poles will be clustered in the left half plane after the transformation. Therefore, the reduced order model is stable.

Finally, we point out that this reduction method is similar to the method proposed by Jonckheere at.el. [60]. Their objective was to reduce the error frequency response as much as possible by minimizing the analytic region of $G(s)$ and making the region at which the poles clustered as big as possible. Then, the region at which the poles are clustered is transformed into left half plane using bilinear transformation. After that, balanced model reduction is performed to obtain a reduced order model in the z-domain. By using the

inverse bilinear transformation, the reduced order model in *s*-domain is obtained. The difference is clear. They assume *G(s)* can have poles any where in *s*-plane except a circle *D*. They transform *D* into the left half plane and the complement region of *D* is transformed to left half plane. In our method, the opposite happens, the left half plane is transformed into certain circle *D*. They choose the bilinear transformation such that it transforms *D* into the right half plane, while we choose the bilinear transformation such that its inverse will transform the error frequency response from a low pass filter into high pass filter. Moreover, their method does not guarantee the stability of the reduced order model while the proposed method produces a stable reduced order method.

The method described in [54] turns out to be a special case of this method at *a=b=0, and c=b=1*.

Examples 3.4

The method is applied to two systems, one is a continuous-time and the other is a discrete-time model. In each example, each parameter of the bilinear transformation is set to zero and the others are changed for several values.

Example 3.1

Consider the following transfer function given in [42]

$$G(s)=\frac{50(s+2)(s+1)(s+0.3)(s+0.1)}{(s+6.0104)(s+0.3009)(s+0.2063)[(s+3.5289)^2+4.2634][(s+0.81)^2+1.3508][(s+0.40230^2+0.0856]}$$

This transfer function is reduced to a third order model using the method developed in this chapter. The parameters of bilinear transformation are chosen as follows:

a) *a* is set to zero and the other parameters are set as follow

i) *b=1, c=1, d=1, 10, 100, 1000, 10000 and 100000.*

ii) *b=1, c=1, 10, 100, 1000, 10000, 100000 and d=1.*

iii) *b=1, 10, 100, 1000, 10000, 100000, c=1 and d=1.*

From figure 3.2a , we see that when *a=0*, the error frequency response is always high pass and it does not depend on the variations of the other parameters. In this case, our method works very good. It converts the error frequency response from low pass to high pass.

b) *b* is set to zero and the other parameters set to as follows

i) *a=1, c=1, d=1, 10, 100, 1000, 10000 and 100000.*

ii) *a=1, c=1, 10, 100, 1000, 10000, 100000 and d=1.*

iii) *a=1, 10, 100, 1000, 10000, 100000, c=1 and d=1.*

Figure 3.2b shows the results of part (i). The error frequency response is the same for all *d*. It is high pass, but the error is not very small at the low frequency region. Changes in *d* does not effect the error response because *d* effects only the factor that the system is multiplyed with in the forward direction of the transformation and in the inverse transformation the system is divided by this factor. Figure 3.2c shows the results of part (ii). The error frequency response is high pass and it decreases in the low frequency region as *c* increases. Figure 3.2d shows the results of part (iii). For *a=1*, the error at the low frequencies is smaller than the error produced by balancing at high frequencies. The error produced by the method is much bigger than the error produced normal balancing. For the other values of *a*, the error looks like the error of balancing except at high frequency it becomes a constant. In this case, balancing results are better than the proposed method.

c) *c* is set to zero and the others as follows

i) *a=1, b=1, d=1, 10, 100, 1000, 10000 and 100000.*

ii) *a=1, b=1, 10, 100, 1000, 10000, 100000 and d=1.*

iii) *a=1, 10, 100, 1000, 10000, 100000, b=1 and d=1.*

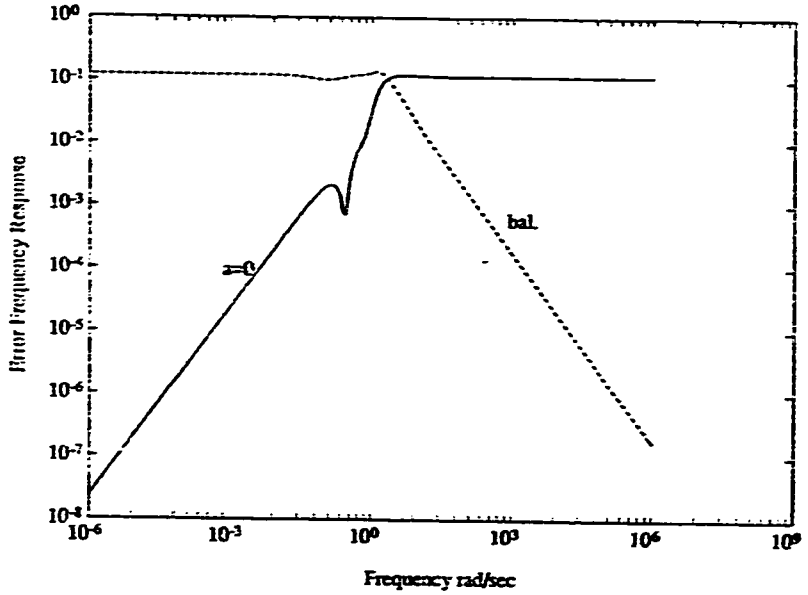

Fig. 3.2a : Error frequency response for $a=0$ and varying the other parameters.

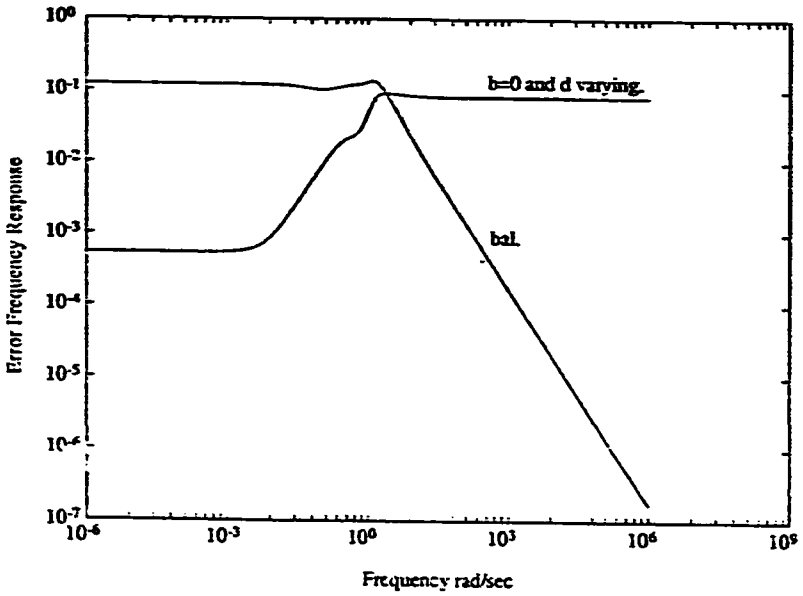

Fig. 3.2b : Error frequency response for $a=1$, $b=0$, $c=1$ and varying d and for $a=1$, $c=1$, $d=0$ and varying b.

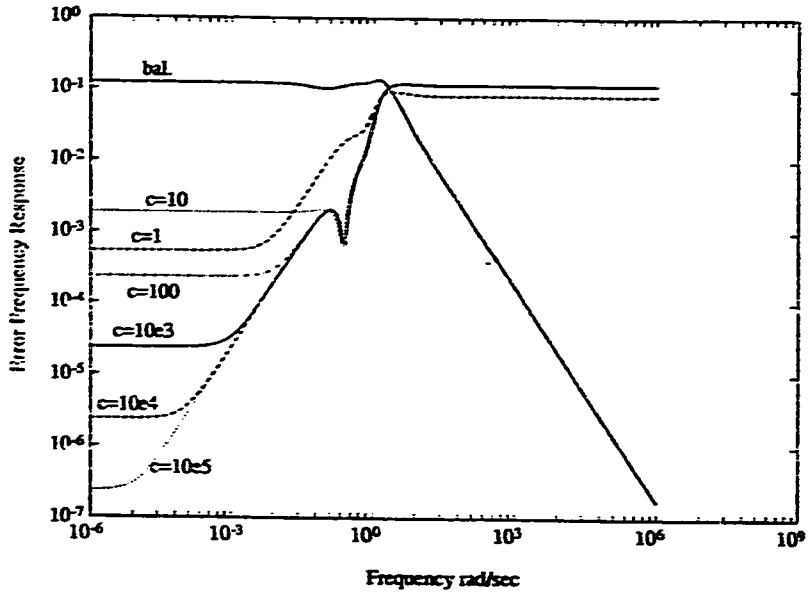

Fig. 3.2c : Error frequency response for *a=1, b=0, d=1* and varying *c* and for *a=1, b=1, d=0* and varying *c*.

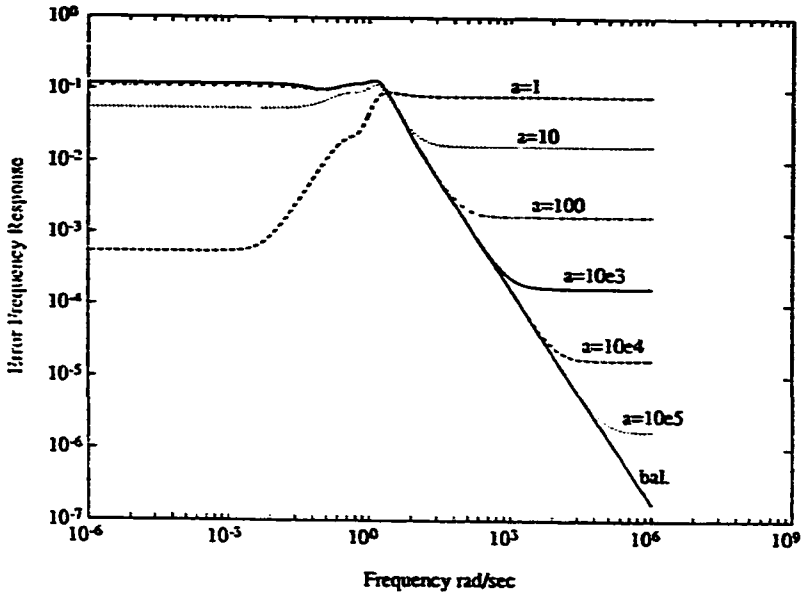

Fig. 3.2d : Error frequency response for *b=0, c=1, d=1* and varying *a* and for *b=1, c=1, d=0* and varying *a*.

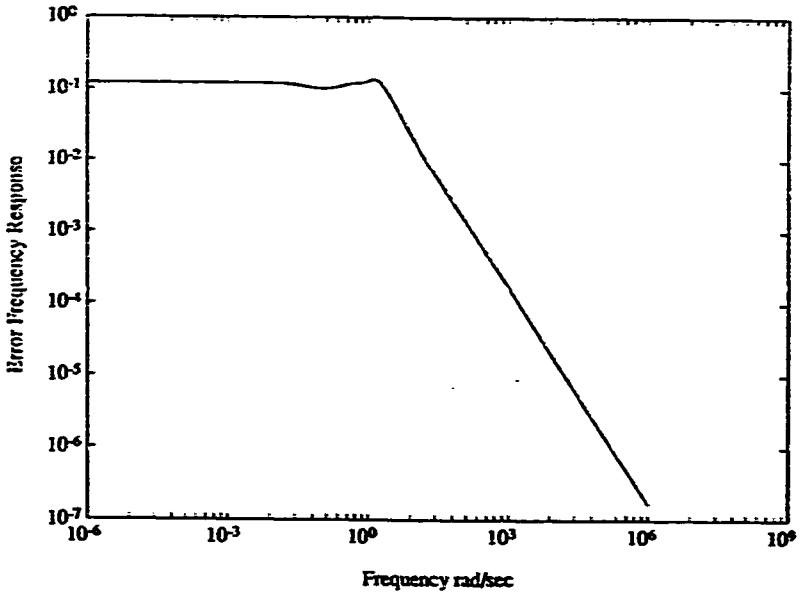

Fig. 3.2e : Error frequency response for $c=0$ and varying the other parameters.

Figure 3.2e shows the results for cases, they are all same as the error frequency response produced by balancing. The variations, in the parameters when $c=0$, do not effect the error frequency response, because the transformation shifts the jw-axis and multiply by a factor and the inverse transformation returns the jw-axis to it original position and divides by the same factor

d) d is set to zero and the others as follows

i) *$a=1, b=1, c=1, 10, 100, 1000, 10000$ and 100000.*

ii) *$a=1, b=1, 10, 100, 1000, 10000, 100000$ and $c=1$.*

iii) *$a=1, 10, 100, 1000, 10000, 100000, b=1$ and $c=1$.*

The results of part (i) are identical to those of part (iii) when $b=0$, the results of part (ii) are identical to those of part (i) when $b=0$, and the results of part (iii) are identical to those of part (ii) when $b=0$.

Example 3.2

This is an example of ninth order boiling water reactor considered in [21] and is discretized using a sampling period of 0.1 second. The A, B, and C matrices of the discretized model are given below.

$$A = \begin{pmatrix} -0.4595 & -0.0283 & 0.0986 & 0.1085 & -0.6198 & 0.0561 & 0.0661 & -0.0001 & -0.0046 \\ -0.0012 & 0.9954 & -0.0104 & -0.0110 & 0.0230 & -0.0055 & -0.0076 & -0.0000 & -0.0001 \\ -0.1118 & 0.2860 & 0.0158 & -1.0337 & 2.1558 & -0.5115 & -0.7108 & -0.0002 & -0.0078 \\ -0.0572 & 0.1464 & -0.4960 & 0.4559 & 1.1033 & -0.2618 & -0.3647 & -0.0001 & -0.0040 \\ 0.0010 & 0.0005 & -0.0016 & 0.0035 & 0.6229 & 0.0019 & -0.0044 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0002 & 0.9738 & 0.0002 & -0.0001 & 0.0260 \\ 0.0017 & 0.0008 & -0.0026 & 0.0061 & -0.0010 & 0.0031 & 0.9907 & 0.0004 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0001 & -0.0000 & 0.0000 & 0.0154 & 0.9048 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0001 & 0.0175 & 0.0000 & 0.0129 & -0.0078 & 0.9890 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0000 & -0.0328 \\ 0.0000 & -0.0006 \\ 0.0000 & -0.0550 \\ 0.0000 & -0.0282 \\ 0.0000 & -0.0003 \\ 0.0000 & -0.0128 \\ -0.0000 & 0.1443 \\ -0.0000 & 0.0011 \\ 0.0000 & 0.0009 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12.247 \end{pmatrix}$$

This model is reduced using the algorithm to a third order reduced model. The the effects of parameters are studied and they are similar to the results get in previous example as shown in figure 3.3a,b,c,d and e.

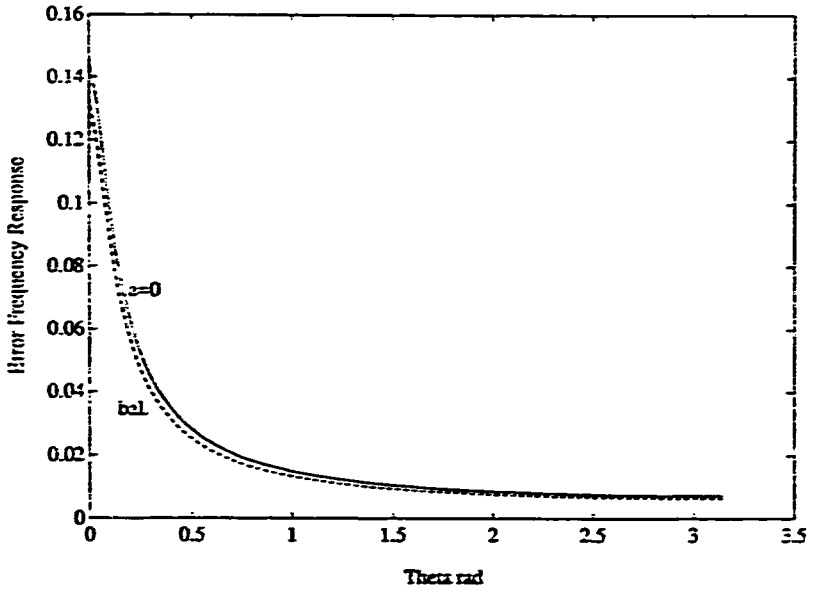

Fig. 3.3a : Error frequency response for *a=0* and varying the other parameters.

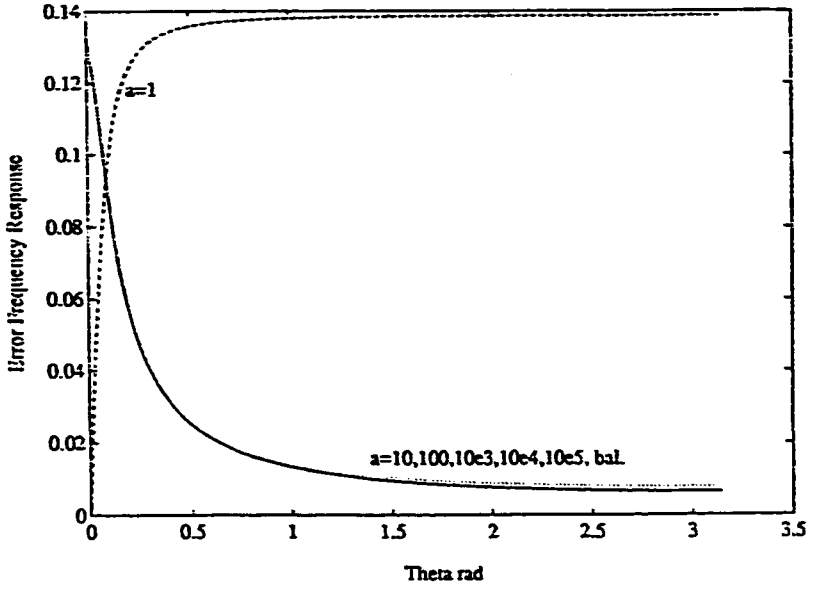

Fig. 3.3b : Error frequency response for *a=1, b=0, c=1* and varying *d* and for *a=1, c=1, d=0* and varying *b*.

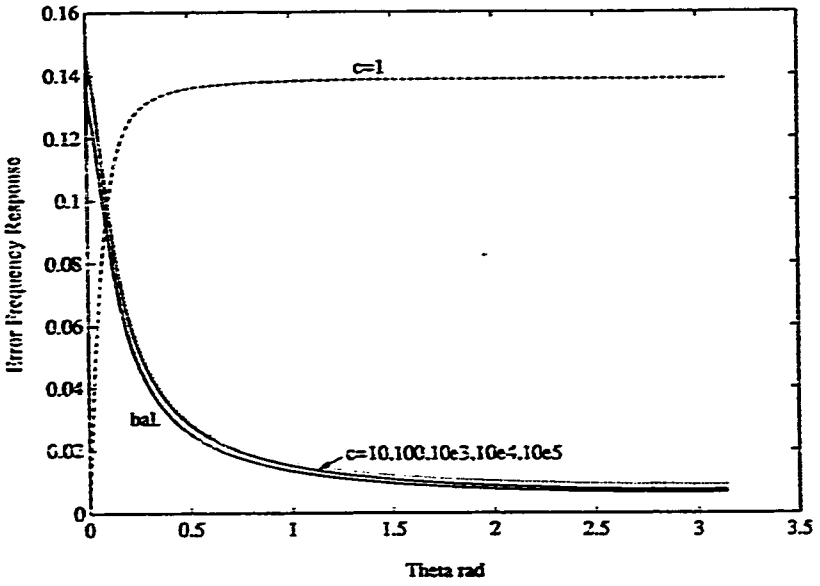

Fig. 3.3c : Error frequency response for $a=1$, $b=0$, $d=1$ and varying c and for $a=1$, $b=1$, $d=0$ and varying c.

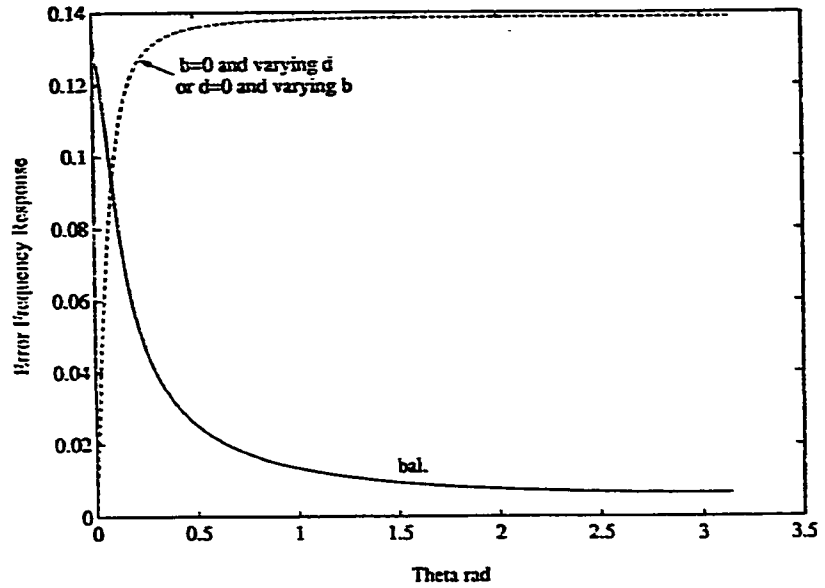

Fig. 3.3d : Error frequency response for $b=0$, $c=1$, $d=1$ and varying a and for $b=1$, $c=1$, $d=0$ and varying a.

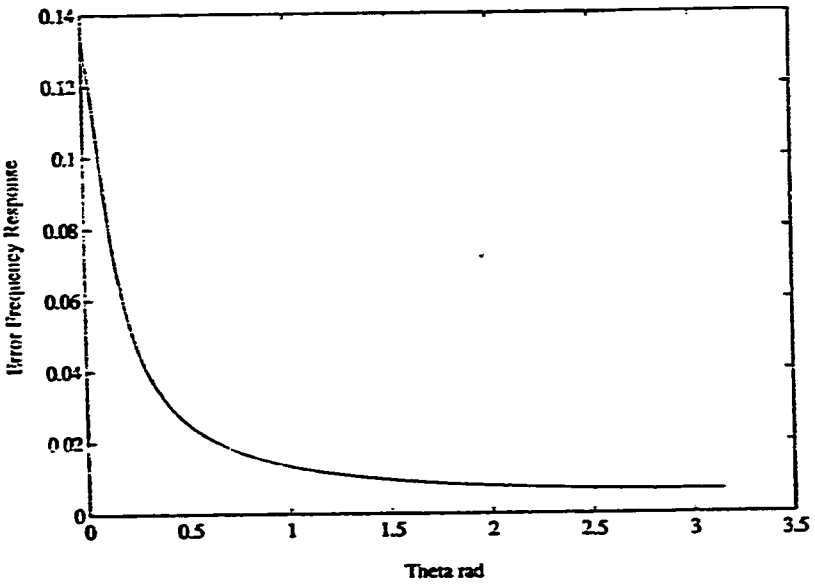

Fig. 3.3e : Error frequency response for $c=0$ and varying the other parameters.

CHAPTER 4

APPROXIMATE BALANCED-TRUNCATION MODEL REDUCTION FOR WEAKLY COUPLED SYSTEMS

4.1 Introduction

Despite all the properties of the balanced-truncated reduced order model, the computations of balanced state space representations for large systems (say order 1000 or more) is still a challenging open problem [39]. Moreover, balancing is intrinsically badly conditioned for systems with some nearly uncontrollable and/or nearly unobservable modes. Since one of the main uses of balanced state space representations is to obtain reduced order models, methods were proposed to derive the reduced order model directly without computing the balancing transformation (Safonov and Chiang [48]). However, this does not eliminate the numerical difficulties associated with large scale systems since the solutions of the two Lyapunov equations (2.4) and (2.5) are needed to drive the reduced order model. Typical examples are in power systems where the interconnection of N generators leads to a state dimension of order $2N$-2 and were it was noted that the Lyapunov equation solver fails to converge for systems with more than 200 generators (Troullinos at el. [56]). Nevertheless, in many situations, experience and intuition indicate how to split a large scale system into a set of simpler weekly coupled subsystems for the purpose of analysis and design. However, the approach that completely neglects coupling in large

scale systems usually leads to unsatisfactory results.

In this chapter, a compromise is sought between the numerical practicability of an approximate balanced-truncation reduced order model and how far is it from an exact balanced-truncation reduced order model. The notation of ε-coupling (week coupling) for large scale systems is used (Kokotovic et al. [38], Kokotovic [37], Jamshidi [31]). The controllability and observability gramians of the system are defined as a power series in the coupling parameter ε.

Approximate balanced representation, in a different context than the one advocated here, were considered by some authors. Jonckheere and Silverman [33] showed that for deformable systems, under some assumptions, balanced-truncation in an asymptotic sense is the same as model truncation as the damping reduced to zero. Jonckheere [32] and Jonckheere and Opendenacher [34] used a parameterization of balanced SISO systems to show the same results. These results were shown for a more general systems by Belloch et al. [9]. However, all the above results were specialized to flexible systems with the damping ratio approaching zero.

This chapter is divided as follows: in section 4.2, weekly coupled (ε-coupled) systems are defined and approximations for the controllability and observability gramians of continuous-time systems will be reviewed. The condition for validity of these approximations, bounds on the norms of the approximation errors, and a measure of the closeness of the approximately balanced representation to the actual one are also reviewed. This sections summarizes results in [2]. Section 4.3 introduces the approximations for the controllability and observability gramians of discrete-time systems, the condition for the validity of these approximations and bounds on the norms of the approximation errors. Finally, section 4.4 gives examples that illustrate the techniques developed in this chapter.

4.2 Approximate Controllability and Observability Gramians For Continuous-Time Systems

In this section, the results of Al-Saggaf [2] will be summarized. In [2] approximate controllability and observability gramiams for continuous-time systems where developed. In next section, these results will be extended to discrete-time systems.

The system of equation (2.1) is said to be ε-coupled if it splits into several independent subsystems when a scalar parameter ε is set to zero (Kokotovic [38]). The system matrices of an ε-coupled system consisting of two subsystems are

$$A=\begin{pmatrix} A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix},\quad B=\begin{pmatrix} B_{11} & \varepsilon B_{12} \\ \varepsilon B_{21} & B_{22} \end{pmatrix},\quad C=\begin{pmatrix} C_{11} & \varepsilon C_{12} \\ \varepsilon C_{21} & C_{22} \end{pmatrix} \tag{4.1}$$

where $A_{11}\in R^{n_1 \times n_1}$, $B_{11}\in R^{n_1 \times m_1}$, $C_{11}\in R^{p_1 \times n_1}$, $A_{22}\in R^{n_2 \times n_2}$, $B_{22}\in R^{n_2 \times m_2}$, $C_{22}\in R^{p_2 \times n_2}$, $n=n_1+n_2$, $m=m_1+m_2$ and $p=p_1+p_2$. The submatrices A_{ij}, B_{ij} and C_{ij} do not depend on ε. Equation (4.1) shows two subsystems, however, generalization to an arbitrary number of subsystems is obvious.

From equations (2.4) and (2.5), we see that both the controllability gramian P and observability gramian Q are analytical in ε. An approximate solutions to equations (2.4) and (2.5) is define as a truncated Maclaurian series in ε [2], i.e.

$$P_a(N)=\sum_{i=0}^{N} P_i\varepsilon^i,\quad Q_a(N)=\sum_{i=0}^{N} Q_i\varepsilon^i \tag{4.2}$$

where

$$P_i=\frac{1}{i!}\frac{d^iP}{d\varepsilon^i},\quad Q_i=\frac{1}{i!}\frac{d^iP}{d\varepsilon^i} \tag{4.3}$$

The following theorem gives a computational procedure for the P_i's. The procedure for the Q_i's follows by duality.

Theorem 4.1 [2]

Partition the P_i's compatibly with the matrices in equation (4.1) as

$$P_i=\begin{pmatrix} P_{i1} & P_{i3} \\ P'_{i3} & P_{i2} \end{pmatrix}, \quad then \quad P_{2i}=\begin{pmatrix} P_{(2i)1} & 0 \\ 0 & P_{(2i)2} \end{pmatrix}, \quad i\geq 0, \quad P_{2i+1}=\begin{pmatrix} 0 & P_{(2i+1)3} \\ P'_{(2i+1)3} & 0 \end{pmatrix}, \quad i\geq 0$$

where

$$A_{11}P_{01}+P_{01}A'_{11}+B_{11}B'_{11}=0$$

$$A_{22}P_{02}+P_{02}A_{22}'+B_{22}B_{22}'=0$$

$$A_{11}P_{(2i)1}+P_{(2i)1}A_{11}'+\{A_{12}P'_{(2i-1)3}+P_{(2i-1)3}A_{12}'\}+S_{(2i)1}=0$$

$$A_{22}P_{(2i)2}+P_{(2i)2}A'_{22}+\{A_{12}P_{(2i-1)3}+P'_{(2i-1)3}A'_{21}\}+S_{(2i)2}=0$$

$$A_{22}P'_{(2i+1)3}+P_{(2i+1)3}A'_{11}+\{A_{21}P_{(2i)1}+P_{(2i)2}A'_{12}\}+S_{(2i+1)3}=0$$

$$S_{21}=B_{12}B'_{12}, \quad S_{22}=B_{21}B'_{21}, \quad S_{(2i)1}=0, \quad S_{(2i)2}=0, \quad i\geq 2$$

$$S_{13}=B_{21}B'_{11}+B_{22}B'_{12}, \quad S_{(2i+1)3}=0, \quad i\geq 1$$

Now let $A=A_0+\varepsilon E$ where the definitions of A_0 and E are obvious from (4.1). Lemma 4.1 shows the behavior of the P_i's.

Lemma 4.1 [2]

Define

$$G=-(A_0\otimes I+I\otimes A_0)^{-1}(E\otimes I+I\otimes I) \tag{4.4}$$

then $\rho(G)<1$ implies $\|P_i\|_F\leq\alpha\|P_2\|_F$; i>2 where $\otimes$ denotes the Kronecker product, $\rho(G)$ is the spectral radius of G, α is a positive constant and $\|P_i\|_F$ is the Frobenius norm of the

matrix P_i.

Lemma 4.2 below gives a condition for the validity of the proposed approximation $P_a(N)$.

Lemma 4.2 [2]

$\lim_{N\to\infty} P_a(N) = P$ if and only if $\rho(\varepsilon G) < 1$

Lemma 4.3 [2]

Let (λ_i, u_i) be an eigenvalue-eigenvector pair of the generalized eigenvalue problem $Eu_i = \lambda_i A_0 u_i$, then $(\lambda_i, u_i \otimes u_i)$ is an eigenvalue-eigenvector pair of the generalized eigenvalue problem

$$(E \otimes I + I \otimes E)x = \mu(A_0 \otimes I + I \otimes A_0)x \tag{4.5}$$

Lemma 4.3 gives an easy check of convergence than Lemma 4.2. This is because G has dimensions of $n^2 \times n^2$ while $A_0^{-1}E$ has dimensions of $n \times n$. Lemma 4.3 shows that $\rho(A_0^{-1}E) < 1$ is a necessary condition for $\rho(G) < 1$. But unfortunately sufficiency is not proven. Al-Saggaf conjectured that $\rho(A_0^{-1}E) < 1$ is also sufficient to have $\rho(G) < 1$. It seems reasonable. Lemma 4.3 shows that for a given system if $\rho(A_0^{-1}(A - A_0)) \geq 1$, then $P_a(N)$ will not converge to P and thus, for any N, it is not a good approximation of P.

Lemma 4.4 [2]

Define $E_N(P) = P - P_a(N)$. If $\rho(\varepsilon G) < 1$, then $E_N(P)$ satisfies the following matrix equation

$$AE_N(P) + E_N(P)A' = \varepsilon^{N+1}(A_0 P_{N+1} + P_{N+1}A_0') \tag{4.6}$$

Theorem 4.2 gives an upper bound for $E_N(P)$, which will help in choosing N.

Theorem 4.2 [2]

$$\| E_N(P) \|_F \leq C_p \varepsilon^{N+1} \tag{4.7}$$

where C_p is a constant independent of N satisfying

$$C_p \le \|H\|_F \|A_0 P_{N+1} + P_{N+1} A_0'\|_F \le 2\alpha \|H\|_F \|P_2\|_F \min\{\|A_0\|_F, \|E\|_F\}$$

and matrix H is defined by

$$AH + HA' + I = 0 \tag{4.8}$$

Now given a large scale system such that $\rho(\overline{G}) < 1$ where

$$\overline{G} = -(A_0 \otimes I + I \otimes A_0)^{-1} (\overline{E} \otimes I + I \otimes \overline{E})$$

and $\overline{E} = A - A_0 = \varepsilon E$ what is an appropriate value for ε? Note that $\overline{G} = \varepsilon G$ where G is defined in Lemma 4.1. The rate of convergence of the approximation is determined by $\rho(\varepsilon G) = \rho(\overline{G})$ and thus the choice of ε might seem unimportant and, light of equation (4.7), might be templed to choose a very small value of ε. However, if a very small value of ε is chosen, then $\rho(G)$ will be greater than one and the P_i's will diverge. Thus we have the situation that $\sum_{i=0}^{N} \varepsilon^i P_i$ is converging and the P_i's is diverging which will lead to numerical difficulties and errors in computing $P_a(N)$ [2]. Thus a value of $\varepsilon < 1$ should be chosen as

$$\rho(\overline{G}) < \varepsilon \le 1 \tag{4.9}$$

This inequality is clear from

$$\rho(G) < 1 \Leftrightarrow \rho\left(\frac{\overline{G}}{\varepsilon}\right) < 1 \Leftrightarrow \rho(\overline{G}) < \varepsilon \text{ and } \varepsilon \le 1.$$

Since $\rho(G)$ is the rate of convergence of P_i's. Hence, the smaller the values of $\rho(G)$ is the better. We know that

$$\rho(G) = \rho\left(\frac{\overline{G}}{\varepsilon}\right) = \frac{1}{\varepsilon}\rho(\overline{G})$$

Therefore, the best value of ε is one, and it makes the computations of $P_a(N)$ much easier.

4.3 Approximate Controllability and Observability Gramians For Discrete-Time Systems

Consider the discrete-time system defined in (2.11). If this system is in the form of equation (4.1), then it is called ε-coupled (weekly coupled) discrete system. From equations (2.14) and (2.15), we can see that both the controllability gramian P and the observability gramian Q are analytical in ε. An approximate solutions to equations (2.14) and (2.15) is defined as truncated Maclaurian series as in (4.2) and (4.3). The computational procedure for P_i's will be given in theorem 4.3 and the procedure for Q_i's follows by duality.

Theorem 4.3

Partition the P_i's compatibly with the matrices in equation (4.1) as

$$P_i=\begin{pmatrix} P_{i1} & P_{i3} \\ P'_{i3} & P_{i2} \end{pmatrix}, \quad then \quad P_{2i}=\begin{pmatrix} P_{(2i)1} & 0 \\ 0 & P_{(2i)2} \end{pmatrix}, \quad i\geq 0, \quad P_{2i+1}=\begin{pmatrix} 0 & P_{(2i+1)3} \\ P'_{(2i+1)3} & 0 \end{pmatrix}, \quad i\geq 0$$

where

$$A_{11}P_{01}A'_{11}-P_{01}+B_{11}B'_{11}=0$$

$$A_{22}P_{02}A'_{22}-P_{02}+B_{22}B'_{22}=0$$

$$A_{11}P_{13}A'_{22}-P_{13}+\{B_{11}B'_{21}+B_{12}B'_{22}+A_{11}P_{01}A'_{21}+A_{12}P_{02}A'_{22}\}=0$$

$$A_{11}P_{21}A'_{11}-P_{21}+\{B_{12}B'_{12}+A_{11}P_{13}A'_{12}+A_{12}P'_{13}A'_{11}+A_{12}P_{02}A'_{12}\}=0$$

$$A_{22}P_{22}A'_{22}-P_{22}+\{B_{21}B'_{21}+A_{22}P_{23}A'_{21}+A_{12}P_{13}A'_{22}+A_{21}P_{01}A'_{21}\}=0$$

$$A_{11}P_{(2i+1)3}A'_{22}-P_{(2i+1)3}+\{A_{11}P_{(2i)1}A'_{21}+A_{12}P_{(2i)2}A'_{22}+A_{12}P'_{(2i-1)3}A'_{21}\}=0, \quad i\geq 1$$

$$A_{11}P_{(2i)11}A'_{11}-P_{(2i)11}+\{A_{11}P_{(2i-1)12}A'_{12}+A_{12}P'_{(2i-1)12}A'_{11}+A_{12}P_{(2i-2)}A'_{12}\}=0,\quad i\geq 2$$

$$A_{22}P_{(2i)22}A'_{22}-P_{(2i)22}+\{A_{22}P'_{(2i-1)12}A'_{12}+A_{12}P_{(2i-1)12}A'_{22}+A_{21}P_{(2i-2)11}A'_{21}\}=0,\quad i\geq 2$$

Proof:

Let

$$A=A_0+\varepsilon E \tag{4.10}$$

and

$$BB'=B_0+\varepsilon B_1+\varepsilon^2 B_2 \tag{4.11}$$

where

$$A_0=\begin{pmatrix}A_{11} & 0\\ 0 & A_{22}\end{pmatrix},\quad E=\begin{pmatrix}0 & A_{12}\\ A_{21} & 0\end{pmatrix}$$

$$B_0=\begin{pmatrix}B_{11}B'_{11} & 0\\ 0 & B_{22}B'_{22}\end{pmatrix},\quad B_1=\begin{pmatrix}0 & B_{11}B'_{21}+B_{12}B'_{22}\\ B_{21}B'_{11}+B_{22}B'_{12} & 0\end{pmatrix}\text{ and }B_2=\begin{pmatrix}B_{12}B'_{12} & 0\\ 0 & B_{21}B'_{21}\end{pmatrix}$$

Assuming the convergence of $P_e(N)$ to P as $N\to\infty$ and substituting (4.10) and (4.11) in (2.14), we get

$$(A_0+\varepsilon E)\left(\sum_{i=0}^{\infty}\varepsilon^i P_i\right)(A'_0+\varepsilon E')-\left(\sum_{i=0}^{\infty}\varepsilon^i P_i\right)+B_0+\varepsilon B_1+\varepsilon^2 B_2=0 \tag{4.12}$$

After simplifications, we get

$$A_0\left(\sum_{i=0}^{\infty}\varepsilon^i P_i\right)A'_0+A_0\left(\sum_{i=0}^{\infty}\varepsilon^{i+1}P_i\right)E'+E\left(\sum_{i=0}^{\infty}\varepsilon^{i+1}P_i\right)A'_0$$
$$+E\left(\sum_{i=0}^{\infty}\varepsilon^{i+2}P_i\right)E'-\left(\sum_{i=0}^{\infty}\varepsilon^i P_i\right)+B_0+\varepsilon B_1+\varepsilon^2 B_2=0 \tag{4.13}$$

The zeroth order term of (4.13) is

$$A_0P_0A'_0-P_0+B_1=0 \tag{4.14}$$

i.e. $$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{01} & P_{03} \\ P'_{03} & P_{02} \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}-\begin{pmatrix} P_{01} & P_{03} \\ P'_{03} & P_{02} \end{pmatrix}+\begin{pmatrix} B_{11}B'_{11} & 0 \\ 0 & B_{22}B'_{22} \end{pmatrix}=0$$

After simple manipulations, we get the following equations

$$A_{11}P_{01}A'_{11}-P_{01}+B_{11}B'_{11}=0 \tag{4.15}$$

$$A_{22}P_{02}A'_{22}-P_{02}+B_{22}B'_{22}=0 \tag{4.16}$$

$$A_{11}P_{03}A'_{22}-P_{03}=0 \tag{4.17}$$

$P_{03}=0$ is a solution of equation (4.17) and it is the only solution if $\lambda(A_{11})\lambda(A_{22})\neq 1$. Thus

$$P_0=\begin{pmatrix} P_{01} & 0 \\ 0 & P_{02} \end{pmatrix}$$

The first order term of (4.13) is

$$A_0P_1A'_0-P_1+B_1+A_0P_0E'+EP_0A'_0=0 \tag{4.18}$$

i.e. $$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{11} & P_{13} \\ P'_{13} & P_{12} \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}-\begin{pmatrix} P_{11} & P_{13} \\ P'_{13} & P_{12} \end{pmatrix}$$

$$+\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{01} & 0 \\ 0 & P_{02} \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} P_{01} & 0 \\ 0 & P_{02} \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}=0$$

After simplifications, we get

$$A_{11}P_{11}A'_{11}-P_{11}=0 \tag{4.19}$$

$$A_{22}P_{12}A'_{22}-P_{12}=0 \tag{4.20}$$

$$A_{11}P_{13}A'_{22}-P_{13}+\{B_{11}B'_{21}+B_{21}B'_{22}+A_{11}P_{01}A'_{21}+A_{21}P_{02}A'_{22}\}=0 \tag{4.21}$$

P_{11} is a solution to equation (4.19) and it is the only solution if $|\lambda(A_{11})|\neq 1$. Similarly, P_{12} is a solution to equation (4.20) and it is the only solution if $|\lambda(A_{22})|\neq 1$. Thus

$$P_1 = \begin{pmatrix} 0 & P_{13} \\ P'_{13} & 0 \end{pmatrix}$$

The second order term of (4.13) is

$$A_0 P_2 A'_0 - P_2 + B_2 + A_0 P_1 E' + E P_1 A'_0 + E P_0 E' = 0 \tag{4.22}$$

i.e.

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{21} & P_{23} \\ P'_{23} & P_{22} \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix} - \begin{pmatrix} P_{21} & P_{23} \\ P'_{23} & P_{22} \end{pmatrix}$$

$$+\begin{pmatrix} B_{12}B'_{12} & 0 \\ 0 & B_{21}B'_{12} \end{pmatrix} + \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} 0 & P_{13} \\ P'_{13} & 0 \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & P_{13} \\ P'_{13} & 0 \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{12} \end{pmatrix} + \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} P_{01} & 0 \\ 0 & P_{02} \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix} = 0$$

After the simplifications, we get

$$A_{11} P_{23} A'_{22} - P_{23} = 0 \tag{4.23}$$

$$A_{11} P_{21} A'_{11} - P_{21} + \{B_{12}B'_{12} + A_{11}P_{13}A'_{12} + A_{12}P'_{13}A'_{11} + A_{12}P_{02}A'_{12}\} = 0 \tag{4.24}$$

$$A_{22} P_{22} A'_{22} - P_{22} + \{B_{21}B'_{21} + A_{22}P_{23}A'_{21} + A_{21}P_{13}A'_{22} + A_{21}P_{01}A'_{21}\} = 0 \tag{4.25}$$

$P_{23} = 0$ is a of equation (4.13) and it is the only solution if $\lambda(A_{11})\lambda(A_{22}) \neq 1$. Thus

$$P_2 = \begin{pmatrix} P_{21} & 0 \\ 0 & P_{22} \end{pmatrix}$$

The i-th order term of equation (4.13) is

$$A_0 P_i A'_0 - P_i + A_0 P_{i-1} E' + E P_{i-1} A'_0 + E P_{i-2} E' = 0, \quad i \geq 3 \tag{4.26}$$

The forms of P_{i-1} and P_{i-2} depend on i. When i is odd, then

$$P_{i-1} = \begin{pmatrix} P_{(i-1)1} & 0 \\ 0 & P_{(i-1)2} \end{pmatrix}, \quad P_{i-2} = \begin{pmatrix} 0 & P_{(i-2)3} \\ P'_{(i-2)3} & 0 \end{pmatrix}$$

and when i is even, then

$$P_{i-1}=\begin{pmatrix} 0 & P_{(i-1)\beta} \\ P'_{(i-1)\beta} & 0 \end{pmatrix}, \quad P_{i-2}=\begin{pmatrix} P_{(i-2)\alpha} & 0 \\ 0 & P_{(i-2)\beta} \end{pmatrix}$$

This can be easily proven by induction. Taking the first case when i is odd and replacing i with $2i+1$, we get

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{(2i+1)\alpha} & P_{(2i+1)\beta} \\ P'_{(2i+1)\beta} & P_{(2i+1)\beta} \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}-\begin{pmatrix} P_{(2i+1)\alpha} & P_{(2i+1)\beta} \\ P'_{(2i+1)\beta} & P_{(2i+1)\beta} \end{pmatrix}$$
$$+\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{(2i)\alpha} & 0 \\ 0 & P_{(2i)\beta} \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix}+\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & P_{(2i-1)\beta} \\ P'_{(2i-1)\beta} & 0 \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix}=0$$

The above equation leads to the following equations

$$A_{11}P_{(2i+1)\alpha}A'_{11}-P_{(2i+1)\alpha}=0 \tag{4.27}$$

$$A_{22}P_{(2i+1)\beta}A'_{22}-P_{(2i+1)\beta}=0 \tag{4.28}$$

$$A_{11}P_{(2i+1)\beta}A'_{22}-P_{(2i+1)\beta}+\{A_{11}P_{(2i)\alpha}A'_{21}+A_{12}P_{(2i)\beta}A'_{22}+A_{12}P'_{(2i-1)\beta}A'_{21}\}=0 \tag{4.29}$$

The trivial solution solves equations (4.27) and (4.28) and it the only solutions if $|\lambda(A_{11})| \neq 1$ and $|\lambda(A_{22})| \neq 1$ respectively. Thus

$$P_{(2i+1)}=\begin{pmatrix} 0 & P_{(2i+1)\beta} \\ P'_{(2i+1)\beta} & 0 \end{pmatrix}, \quad i \geq 1$$

Finally, taking i even and replacing it with $2i$ in equation (4.26), we get

$$\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} P_{(2i)\alpha} & P_{(2i)\beta} \\ P'_{(2i)\beta} & P_{(2i)\beta} \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}-\begin{pmatrix} P_{(2i)\alpha} & P_{(2i)\beta} \\ P'_{(2i)\beta} & P_{(2i)\beta} \end{pmatrix}$$
$$+\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} 0 & P_{(2i-1)\beta} \\ P'_{(2i-1)\beta} & 0 \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} 0 & P_{(2i-1)3} \\ P'_{(2i-1)3} & 0 \end{pmatrix}\begin{pmatrix} A'_{11} & 0 \\ 0 & A'_{22} \end{pmatrix}$$

$$+\begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}\begin{pmatrix} P_{(2i-2)1} & 0 \\ 0 & P_{(2i-2)2} \end{pmatrix}\begin{pmatrix} 0 & A'_{21} \\ A'_{12} & 0 \end{pmatrix}=0$$

After simplifications we get

$$A_{11}P_{(2i)3}A'_{22}-P_{(2i)3}=0 \tag{4.30}$$

$$A_{11}P_{(2i)1}A'_{11}-P_{(2i)1}+\{A_{11}P_{(2i-1)3}A'_{12}+A_{12}P'_{(2i-1)3}A'_{11}+A_{12}P_{(2i-2)2}A'_{12}\}=0 \tag{4.31}$$

$$A_{22}P_{(2i)2}A'_{22}-P_{(2i)2}+\{A_{22}P'_{(2i-1)3}A'_{21}+A_{21}P_{(2i-1)3}A'_{22}+A_{21}P_{(2i-2)2}A'_{21}\}=0 \tag{4.32}$$

$P_{(2i)3}=0$ is a solution of equation (4.30) and it is the only solution if $\lambda(A_{11})\lambda(A_{22})\neq 1$. Thus

$$P_{2i}=\begin{pmatrix} P_{(2i)1} & 0 \\ 0 & P_{(2i)2} \end{pmatrix},\quad i\geq 2$$

The following lemma shows the behavior of the P_i's.

Lemma 4.5

Define

$$G=\begin{pmatrix} 0 & I \\ -(A_0\otimes A_0-I)^{-1}(A_0\otimes E-E\otimes A_0) & -(A_0\otimes A_0-I)^{-1}(E\otimes E) \end{pmatrix} \tag{4.33}$$

then $\rho(G)<1$ implies $\|P_i\|_F\leq\alpha\sqrt{\|P_1\|_F^2+\|P_2\|_F^2}$; $i\geq 3$ where $\otimes$ denotes the Kronecker product, $\rho(G)$ is the spectral radius of G, α is a positive constant and $\|P_i\|_F$ is the Frobenius norm of the matrix P_i.

Proof

Consider the i-th order term of equation (4.13) given by (4.26)

$$A_0P_iA'_0-P_i+A_0P_{i-1}E'+EP_{i-1}A_0+EP_{i-2}E'=0$$

Defining $Z_i = \mathrm{Vec}(P_i)$ as a vector of size n^2 formed by stacking the columns of the matrix P_i, then the equation (4.26) is equivalent to the following linear system [25]

$$(A_0 \otimes A_0 - I)Z_i + (A_0 \otimes E + E \otimes A_0)Z_{i-1} + (E \otimes E)Z_{i-2} = 0 \qquad (4.34)$$

This equation can be transformed into the following second order difference system

$$w(i) = Gw(i-1), \quad i \geq 3$$

where

$$w(i) = \begin{pmatrix} Z_{i-1} \\ Z_i \end{pmatrix}$$

If $\rho(G) < 1$, then $\|w(i)\|_F \leq \|w(2)\|_F$ where $\|w(i)\|_F$ denotes the Euclidean norm of the vector $w(i)$. Using the definition of $w(i)$, the above equation implies

$$\|Z_{i-1}\|^2 + \|Z_i\|^2 \leq \alpha^2\{\|Z_1\|^2 + \|Z_2\|^2\}$$

Therefore

$$\|Z_i\| \leq \alpha\sqrt{\|Z_1\|^2 + \|Z_2\|^2}$$

Since $Z_i = \mathrm{Vec}(P_i)$, it follows that

$$\|P_i\|_F \leq \alpha\sqrt{\|P_1\|_F^2 + \|P_2\|_F^2}\ ; i \geq 3 \qquad (4.35)$$

Lemma 4.6 below gives a condition for the validity of the proposed approximation $P_a(N)$.

Lemma 4.6

$$\lim_{N \to \infty} P_a(N) = P \text{ if and only if } \rho(\varepsilon G) < 1 \qquad (4.36)$$

Proof

Consider the infinite sum $\sum_{i=0}^{\infty} \varepsilon^i w(i)$ where the $w(i)$'s are as defined in the prove of the previous lemma. Now

$$\sum_{i=0}^{\infty} \varepsilon^i w(i) = w(0) + \varepsilon w(1) + \varepsilon^2 w(2) + \varepsilon^3 \left(\sum_{i=3}^{\infty} (\varepsilon G)^{i-3} \right) w(3)$$

This series converges if and only if $\rho(\varepsilon G) < 1$. In this case $w = \sum_{i=0}^{\infty} \varepsilon^i w(i)$ where $w = \begin{pmatrix} \varepsilon Z \\ Z \end{pmatrix}$

and Z= Vec(P). Thus $P = \lim_{N \to \infty} P_a(N)$.

The value of $\rho(\varepsilon G)$ is equal to the value of $\rho(G)$ when $\varepsilon = 1$. This means the convergence of the algorithm completely depends only the structure of the system. Theorem 4.4 gives a bound for this approximation. This bound helps in choosing N. The following lemma is needed in the sequel and will used in the prove of the theorem.

Lemma 4.7

Let $E_N(P) = P - P_a(N)$. If $\rho(\varepsilon G) < 1$, then $E_N(P)$ satisfies the following matrix equation

$$A E_N(P) A' - E_N(P) = \varepsilon^{N+1} (A_0 P_{N+1} A'_0 - P_{N+1} - \varepsilon E P_N E') \tag{4.37}$$

Proof

$$E_N(P) = P - P_a(N) = \sum_{i=0}^{\infty} \varepsilon^i P_i - \sum_{i=0}^{N} \varepsilon^i P_i = \sum_{i=N+1}^{\infty} \varepsilon^i P_i$$

Thus

$$A E_N(P) A' - E_N(P) = \sum_{i=N+1}^{\infty} \varepsilon^i (A P_i A' - P_i)$$

$$= \sum_{i=N+1}^{\infty} \varepsilon^i [A_0 P_i A'_0 - P_i + \varepsilon (A_0 P_i E' + E P_i A'_0) + \varepsilon^2 (E P_i E')]$$

$$= \sum_{i+N+1}^{\infty} \varepsilon^i [A_0 P_i A'_0 - P_i + \varepsilon (A_0 P_{i+1} A'_0 - P_{i+1} + E P_{i-1} E') + \varepsilon^2 (E P_i E')]$$

$$= \sum_{i=N+1}^{\infty} \varepsilon^i [A_0(P_i - \varepsilon P_{i+1})A'_0 - (P_i - \varepsilon P_{i+1}) + \varepsilon E(P_{i-1} - \varepsilon P_i)E']$$

$$= \varepsilon^{N+1}(A_0 P_{N+1} A'_0 - P_{N+1}) - \varepsilon^{N+2} E P_N E'$$

The equality in the third step follows by using equation (4.26).

Theorem 4.4 below gives a bound on the norm of the approximation error $E_N(P)$ as a function of N.

Theorem 4.4

$$\| E_p(N) \|_F \leq C_p \varepsilon^{N+1} \tag{4.38}$$

where C_p is a constant independent of N satisfying

$$C_p \leq \| H \|_F \| (A_0 P_{N+1} A'_0 - P_{N+1}) - \varepsilon E P_N E' \|_F \leq \alpha \| H \|_F \| P_2 \|_F \left[\| A_0 \|_F^2 + \| E \|_F^2 + 1 \right]$$

and the matrix H is defined by

$$AH + HA' + I = 0 \tag{4.39}$$

Proof

Subtracting equation (4.37) from the reachability gramian equation (2.14), we get

$$AP_e A' - P_e + \{BB' + \varepsilon^{N+1}(A_0 P_{N+1} A'_0 - P_{N+1} - \varepsilon E P_N E')\} = 0 \tag{4.40}$$

Now treating equation (2.14) as a perturbed version of equation (4.40) and using the results on the sensitivity of the stable discrete-time Lyapunov equation given by Gahinet et al. [61], We get

$$\| E_N(P) \|_F \leq \varepsilon^{N+1} \| H \|_F \| A_0 P_{N+1} A'_0 - P_{N+1} - \varepsilon E P_N E' \|_F \tag{4.41}$$

Using the triangle inequality and (4.35), (4.41) becomes after simplifications and using the fact that $\varepsilon < 1$

$$\| E_N(P) \|_F \leq \varepsilon^{N+1} \alpha \| H \|_F \| P_2 \|_F \left[\| A_0 \|_F^2 + \| E \|_F^2 + 1 \right]$$

This completes the proof.

Similarly defining $E_N(Q) = Q - Q_e(N)$ and using duality we can show that

$$\| E_N(Q) \| \leq C_Q \varepsilon^{N+1} \tag{4.42}$$

where C_Q is a constant independent of N defined in a similar way as C_P.

Finally, one may ask: what is an appropriate value for ε? to answer this equation, let us define $\overline{G}$ as $\overline{G} = \varepsilon G$. Then, it follows that

$$\varepsilon \rho(G) = \rho(\overline{G}) \tag{4.43}$$

Thus $\rho(\overline{G})$ does not depend on ε is a constant independent of ε and a property of the system. If we choose a very small value for ε, then $\rho(G)$ will become greater than one and the P_i's will diverge. From lemma 4.5, we have

$$\rho(G) < 1 \quad \Leftrightarrow \quad \frac{1}{\varepsilon}\rho(\overline{G}) < 1 \quad \Leftrightarrow \quad \rho(\overline{G}) < \varepsilon \leq 1 \tag{4.44}$$

The inequality (4.44) gives the bounds for ε; namely ε should be less than or equal one otherwise ε^i will diverge, and ε should be greater than $\rho(\overline{G})$ otherwise P_i's will diverge. From (4.43), if ε is big, $\rho(G)$ becomes small. This means the P_i's will converge fast. Therefore, we should make ε as big as possible. Thus, ε should be equal to one. By making $\varepsilon = 1$, the computations for ε^i will be eliminated and P_i's will converge fast implying few of iterations.

4.3 Examples

In this section, Three examples will be used to illustrate the properties of the algorithm developed in this chapter. The dependance on N and ε will also be studies in these exam-

ples.

Example 4.1

A twelve order model of twelve plate absorption system [42] was discretized with a sampling period of 1 second. The state space representation of the system is partitioned as

$$A_{11}=\begin{pmatrix} 0.2092 & 0.1257 & 0.0388 & 0.0081 & 0.0013 & 0.0002 \\ 0.1068 & 0.2421 & 0.1325 & 0.0398 & 0.0082 & 0.0013 \\ 0.0280 & 0.1126 & 0.2430 & 0.1327 & 0.0399 & 0.0082 \\ 0.0049 & 0.0288 & 0.1127 & 0.2430 & 0.1327 & 0.0399 \\ 0.0007 & 0.0050 & 0.0288 & 0.1127 & 0.2430 & 0.1327 \\ 0.0001 & 0.0007 & 0.0050 & 0.0288 & 0.1127 & 0.2430 \end{pmatrix}$$

$$A_{12}=\begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0002 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0013 & 0.0002 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0082 & 0.0013 & 0.0002 & 0.0000 & 0.0000 & 0.0000 \\ 0.0399 & 0.0082 & 0.0013 & 0.0002 & 0.0000 & 0.0000 \\ 0.1327 & 0.0399 & -0.0082 & 0.0013 & 0.0002 & 0.0000 \end{pmatrix}$$

$$A_{21}=\begin{pmatrix} 0.0000 & 0.0001 & 0.0007 & 0.0050 & 0.0288 & 0.1127 \\ 0.0000 & 0.0000 & 0.0001 & 0.0007 & 0.0050 & 0.0288 \\ 0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0007 & 0.0050 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0001 & 0.0007 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0001 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}$$

$$A_{22}=\begin{pmatrix} 0.2430 & 0.1327 & 0.0399 & 0.0082 & 0.0013 & 0.0002 \\ 0.1127 & 0.2430 & 0.1327 & 0.0399 & 0.0082 & 0.0013 \\ 0.0288 & 0.1127 & 0.2430 & 0.1327 & 0.0398 & 0.0081 \\ 0.0050 & 0.0288 & 0.1127 & 0.2430 & 0.1325 & 0.0388 \\ 0.0007 & 0.0050 & 0.0288 & 0.1126 & 0.2421 & 0.1257 \\ 0.0001 & 0.0007 & 0.0049 & 0.0280 & 0.1068 & 0.2092 \end{pmatrix}$$

$$B_{11}=\begin{pmatrix}0.2653 & 0\\ 0.0523 & 0\\ 0.0079 & 0\\ 0.0010 & 0\\ 0.0001 & 0\\ 0.0000 & 0\end{pmatrix},\quad B_{12}=\begin{pmatrix}0 & 0.0000\\ 0 & 0.0000\\ 0 & 0.0001e-5\\ 0 & 0.0014e-5\\ 0 & 0.0203e-5\\ 0 & 0.2634e-5\end{pmatrix}$$

$$B_{21}=\begin{pmatrix}0.6058e-6 & 0\\ 0.0396e-6 & 0\\ 0.0023e-6 & 0\\ 0.0001e-6 & 0\\ 0.0000 & 0\\ 0.0000 & 0\end{pmatrix},\quad B_{22}=\begin{pmatrix}0 & 0.0000\\ 0 & 0.0003\\ 0 & 0.0026\\ 0 & 0.0179\\ 0 & 0.1006\\ 0 & 0.4338\end{pmatrix}$$

$$C_{11}=\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\end{pmatrix},\quad C_{12}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$C_{21}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix},\quad C_{22}=\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

The condition (4.36) in lemma 4.6 should be first checked to make sure the algorithm will converge. The value of $\rho(\overline{G})$ is 0.1984 which means the algorithm will converge for any value of ε. The value of ε will be chosen according to (4.44) to guarantee the convergence of P_i's and Q_i's. It should satisfy the following

$$0.1984 < \varepsilon \leq 1$$

The following values of ε were used

$$\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$$

However, it was noted that the rate of convergence of the P_i's and Q_i's were the same for all values of ε as can be seen from figures 4.1a and 4.2b respectively. This is due to the fact that the coupling in this system is very weak. Therefore, The zeroth order approximation is a very good approximation and the contribution of the higher order terms is negligible.

Thus ε has almost no effect on the computations of $P_e(N)$ and $Q_e(N)$. Figures 4.1a and 4.1bshow a plot of $\|P_i\|_F/\|P\|_F$ and $\|Q_i\|_F/\|Q\|_F$ respectively where P and Q are respectively the controllability and observability gramains of the system. From these plots we see clearly that $\|P_0\|_F = \|P\|_F$ and $\|Q_0\|_F = \|Q\|_F$ indicating the same conclusion regarding the effect of ε on the approximation. Figures 4.1c and 4.1d show respectively a plot of $\|E_N(P)\|_F/\|P\|_F$ and $\|E_N(Q)\|_F/\|Q\|_F$ versus N. Again these plots indicate that the zeroth order approximation is a good approximation for both P and Q and the approximation becomes exact for values of $N \geq 3$.

Finally, figure 4.1e shows a plot of the reduction errors for balanced-truncated and approximation balanced-truncated model reduction methods for several values of N. It is seen from this figure that the two methods give the same error for values of $N \geq 3$.

Example 4.2

The system chosen in this example is not weakly coupled. However, it still satisfies the condition of equation (4.36). Therefore, the approximation algorithm can be used.

Consider the seventh-order model of single machine-infinite bus power system [42]. This system was discretized with a sampling period of 0.25 seconds and partitioned as

$$A_{11} = \begin{pmatrix} 0.9949 & 0.2444 & -0.0062 & -0.0011 \\ -0.0384 & 0.9542 & 0.0495 & -0.0056 \\ -0.1029 & -0.0126 & 0.7994 & 0.0348 \\ -0.2114 & -0.0386 & 0.6330 & 0.0278 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} -0.0006 & -0.0001 & 0.0001 \\ -0.0027 & -0.0004 & 0.0003 \\ 0.0012 & 0.0001 & 0.0014 \\ 0.0011 & 0.0001 & 0.0011 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} -0.2321 & 0.0468 & -0.0409 & -0.0018 \\ -0.4944 & -0.1148 & 0.3781 & 0.0168 \\ -0.3087 & -0.0734 & -0.0532 & -0.0021 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0.0014 & -0.0001 & -0.0001 \\ 0.0009 & 0.0001 & 0.0006 \\ 0.0006 & 0.0000 & -0.0001 \end{pmatrix}$$

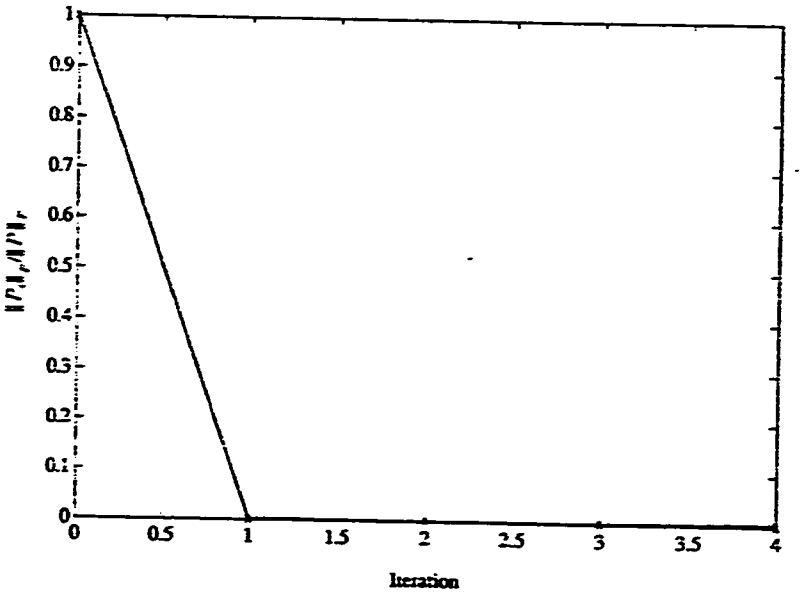

Fig. 4.1a : The normalized Frobenius norm of P_i.

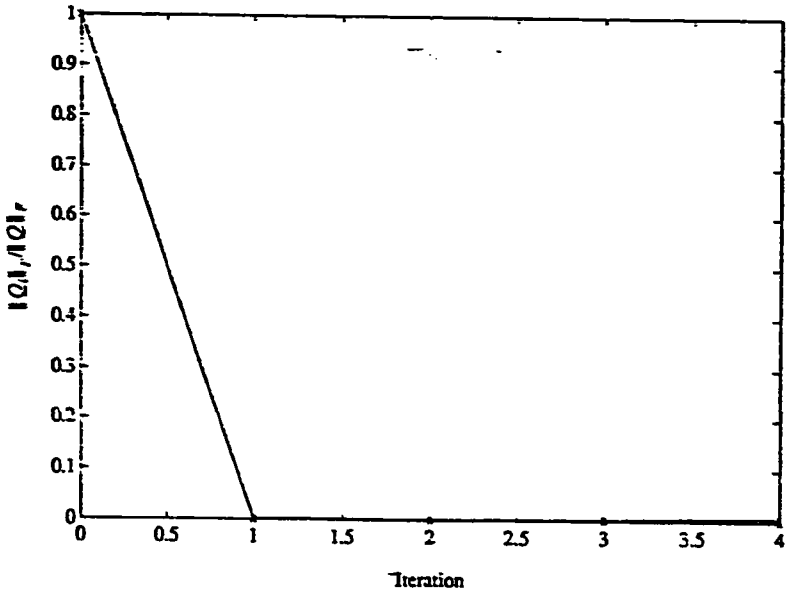

Fig. 4.1b : The normalized Frobenius norm of Q_i.

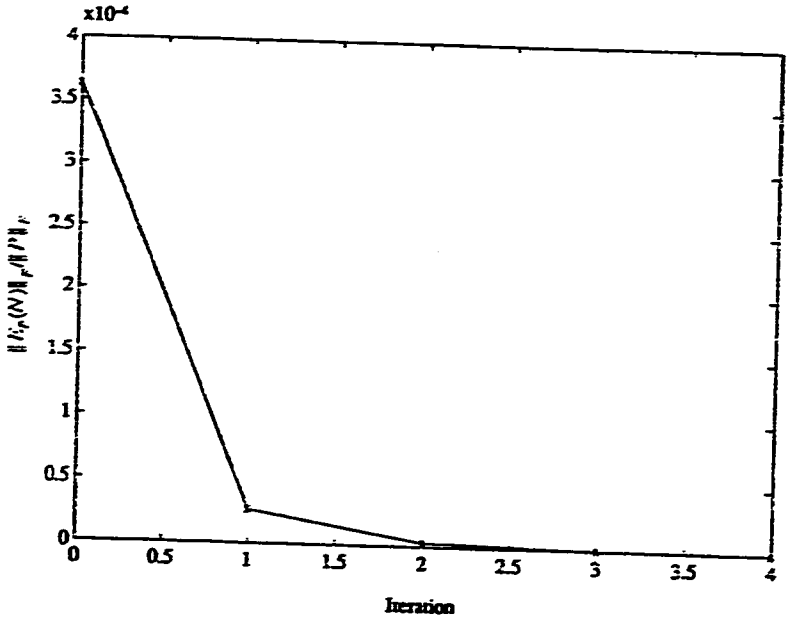

Fig. 4.1c : The normalized Frobenius norm of $E_P(N)$.

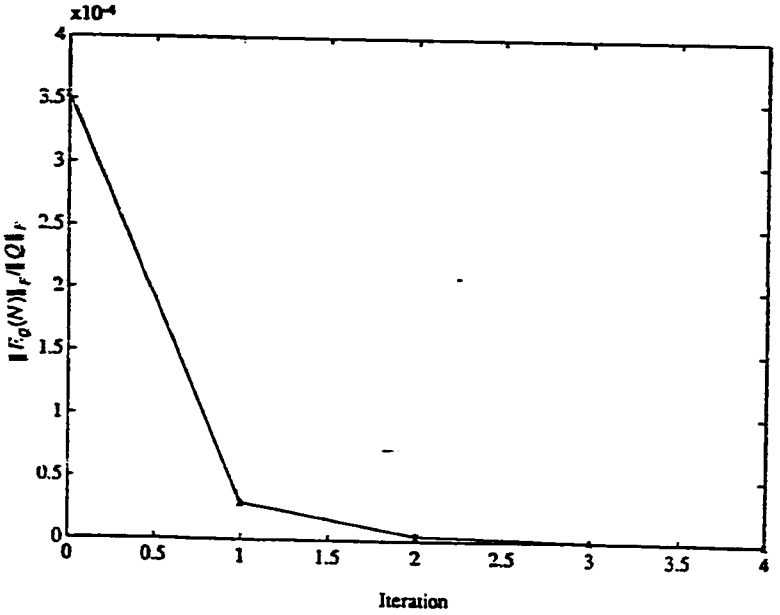

Fig. 4.1d : The normalized Frobenius norm of $E_Q(N)$.

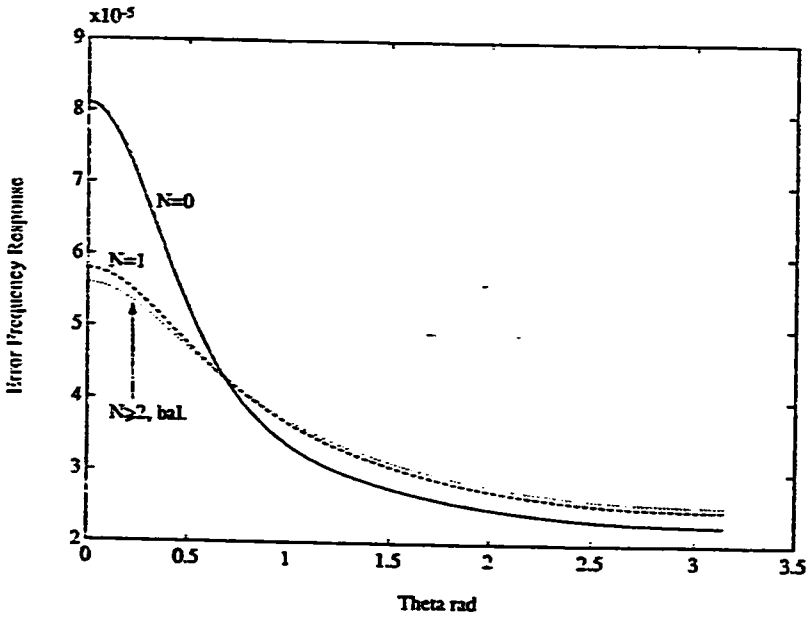

Fig. 4.1e : Error frequency response.

$$B_{11}=\begin{pmatrix}-11.3888\\ 0.9949\\ 7.2007\\ 7.1947\end{pmatrix},\ B_{12}=\begin{pmatrix}2.2860\\ -0.0049\\ -1.4505\\ -1.4475\end{pmatrix},\ B_{21}=\begin{pmatrix}2.3474\\ 8.4932\\ 3.1212\end{pmatrix},\ B_{22}=\begin{pmatrix}-0.4686\\ -1.7071\\ -0.6256\end{pmatrix}$$

$C_{11}=(0\quad 0\quad 0\quad 1),\ C_{12}=(0\quad 0\quad 0),\ C_{21}=(0\quad 0\quad 0\quad 0),\ C_{22}=(0\quad 0\quad 1)$

The value of $\rho(\overline{G})$ is 0.1007 and the approximation algorithm was applied for different values of ε and N.

Figures 4.2a and 4.2b show respectively plots of $\|P_i\|_F/\|P\|_F$ and $\|Q_i\|_F/\|Q\|_F$ for different values of ε. From these plots, as expected, increasing the value of ε, leads to faster rates of convergence of the P_i's and Q_i's. Figures 4.2c and 4.2d show respectively a plot of $\|E_P(N)\|_F/\|P\|_F$, and $\|E_Q(N)\|_F/\|Q\|_F$.

P_i and Q_i converges at the eighth iteration as can be seen in figures 4.1a and 4.1b. However, $E_P(N)$ and $E_Q(N)$ converge to zero in second iteration as can be seen from figures 4.2c and 4.2d. The convergence of $E_P(N)$ and $E_Q(N)$ is much important than the convergence of P_i and Q_i, because if $E_P(N)$ and $E_Q(N)$ converge to zero, $P_a(N)$ and $Q_a(N)$ will converge to P and Q. Figures 4.2e through 4.2i show the error frequency response for different values of N. For $N\geq 7$, the error frequency response for the approximate balanced-truncation model reduction matches the error frequency response obtained using balanced-trucation model reduction. The convergence is not very fast because the coupling between the subsystems is strong.

Example 4.3

A six-plate gas-obsorber system described by Bahnaswi et al. [8] is discretized with a sampling period of 0.8 seconds and partitioned as

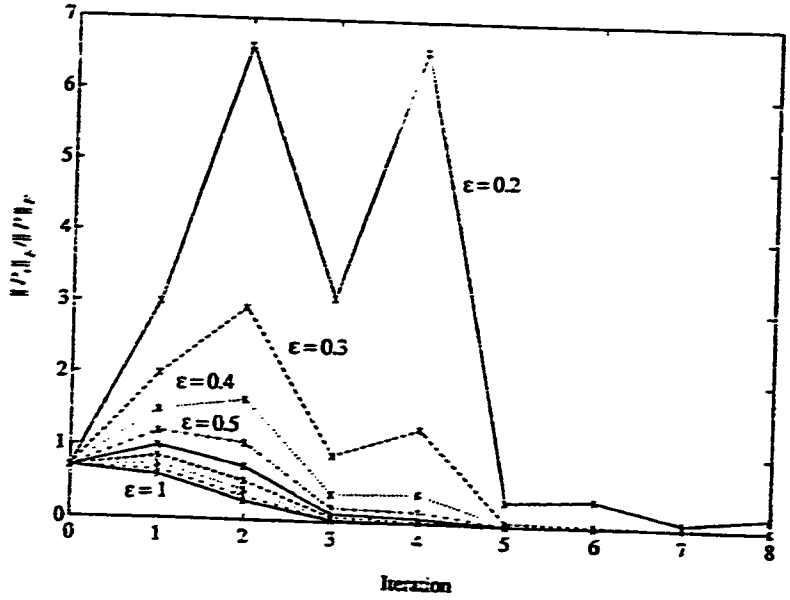

Fig. 4.2a : The normalized Frobenius norm of P_i.

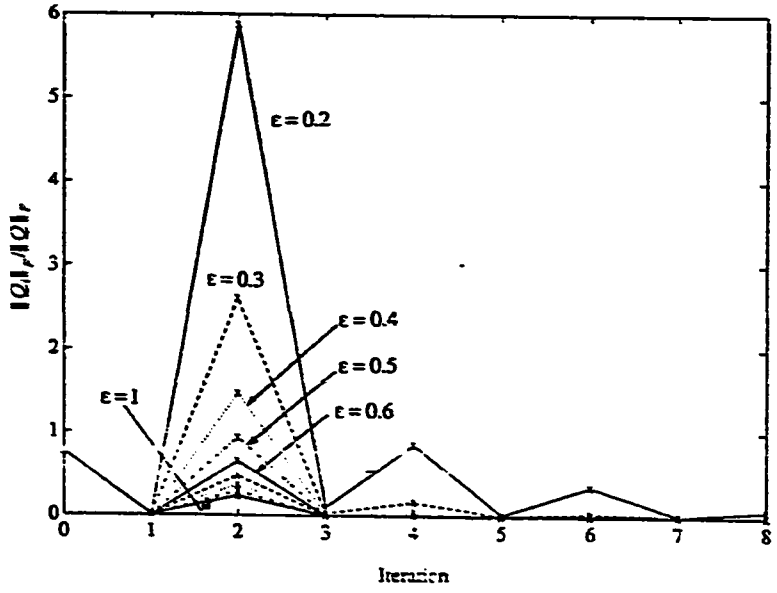

Fig. 4.2b : The normalized Frobenius norm of Q_i.

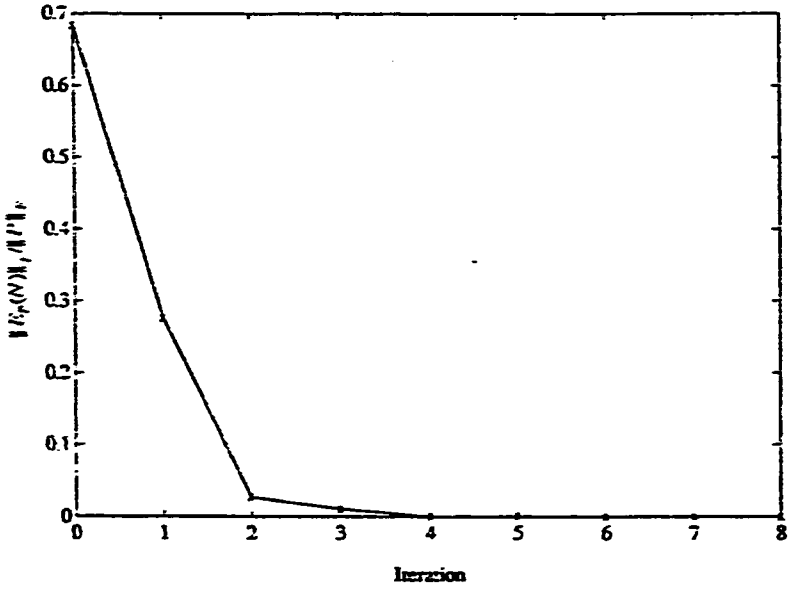

Fig. 4.2c : The normalized Frobenius norm of $E_P(N)$.

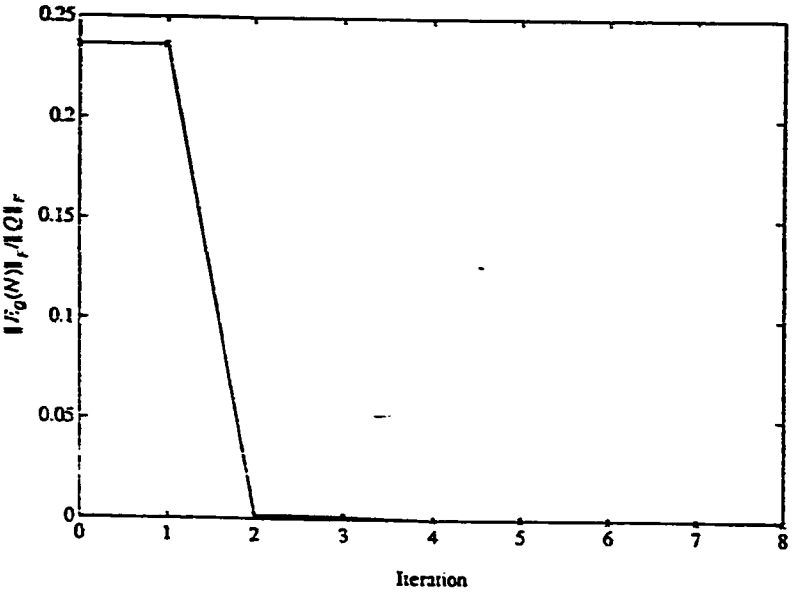

Fig. 4.2d : The normalized Frobenius norm of $E_Q(N)$.

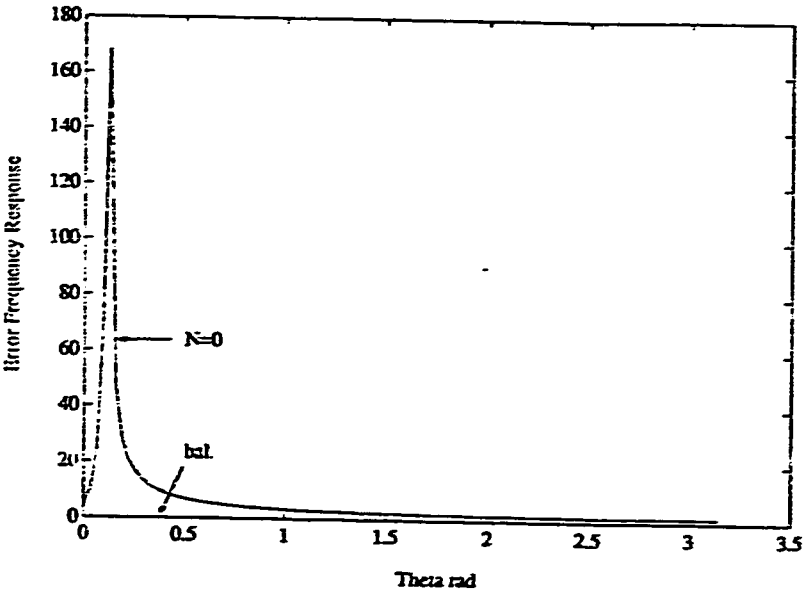

Fig. 4.2e : Error frequency response for N=0 and balancing.

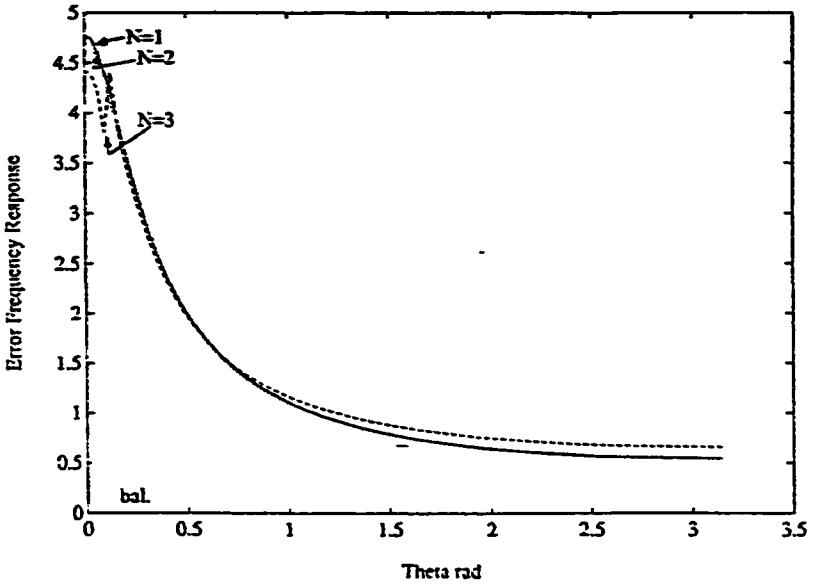

Fig. 4.2f : Error frequency response for N=1,2,3 and balancing.

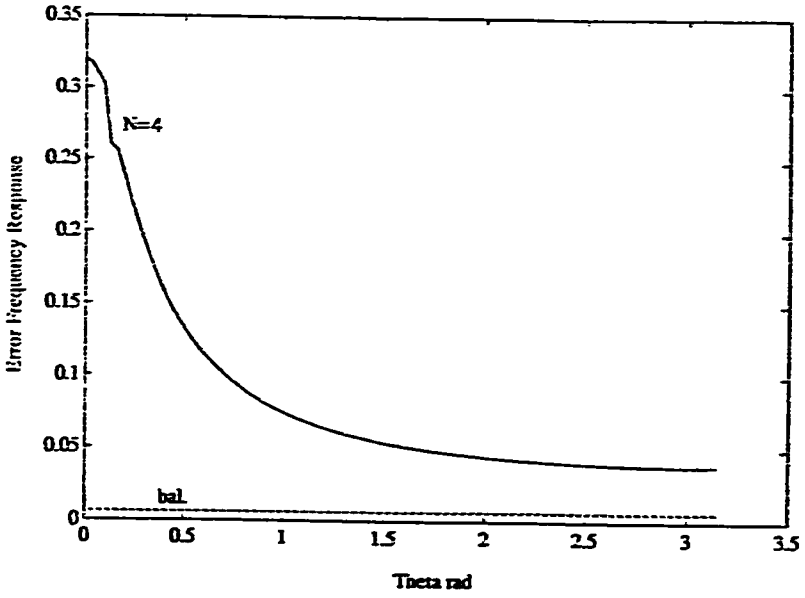

Fig. 4.2g : Error frequency response for N=4 and balancing.

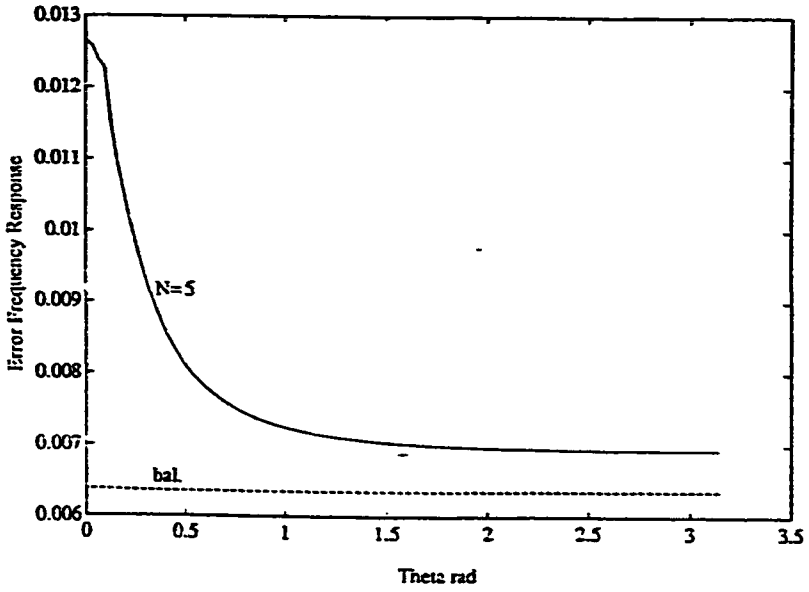

Fig. 4.2h : Error frequency response for N=5 and balancing.

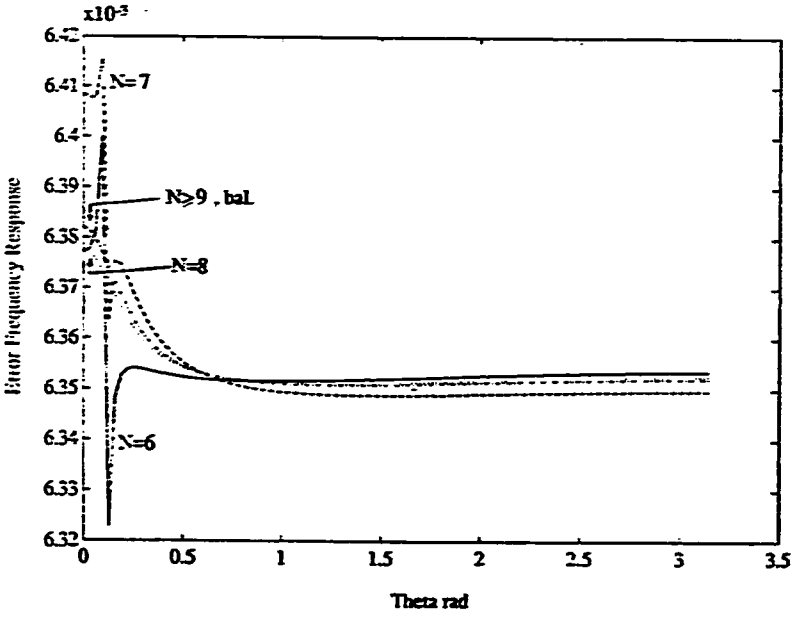

Fig. 4.2i : Error frequency response for $N \geq 6$ and balancing.

$$A_{11}=\begin{pmatrix}0.4355 & 0.2133 & 0.0531\\ 0.1810 & 0.4806 & 0.2208\\ 0.0383 & 0.1874 & 0.4814\end{pmatrix},\ A_{12}=\begin{pmatrix}0.0089 & 0.0011 & 0.0001\\ 0.0541 & 0.0090 & 0.0011\\ 0.2209 & 0.0541 & 0.0089\end{pmatrix}$$

$$A_{21}=\begin{pmatrix}0.0054 & 0.0389 & 0.1875\\ 0.0006 & 0.0055 & 0.0389\\ 0.0000 & 0.0006 & 0.0054\end{pmatrix},\ A_{22}=\begin{pmatrix}0.4814 & 0.2208 & 0.0531\\ 0.1874 & 0.4806 & 0.2133\\ 0.0383 & 0.1810 & 0.4355\end{pmatrix}$$

$$B_{11}=\begin{pmatrix}0.2872\\ 0.0525\\ 0.0069\end{pmatrix},\ B_{12}=\begin{pmatrix}0.0000\\ 0.0002\\ 0.0019\end{pmatrix},\ B_{21}=\begin{pmatrix}0.7023e-3\\ 0.0583e-3\\ 0.0041e-3\end{pmatrix},\ B_{22}=\begin{pmatrix}0.0157\\ 0.1011\\ 0.4698\end{pmatrix}$$

$C_{11}=(1\ \ 0\ \ 0)$, $C_{21}=(0\ \ 0\ \ 0)$, $C_{12}=(0\ \ 0\ \ 0)$ $C_{22}=(1\ \ 0\ \ 0)$

The value of $\rho(\overline{G})$ is 0.6442 which means that we can choose ε in the range: $0.6442 < \varepsilon \leq 1$. The following values of ε were used $\{0.7, 0.8, 0.9, 1.0\}$.

Figures 4.3a and 4.3b show respectively plots of $\|P_i\|_F/\|P\|_F$ and $\|Q_i\|_F/\|Q\|_F$ for different values of ε. The plots indicate a faster rate of convergence to zero for higher values of ε. Figures 4.3c and 4.3d show respectively a plot of $\|E_P(N)\|_F/\|P\|_F$ and $\|E_Q(N)\|_F/\|Q\|_F$.

All figures show that as N increases the approximation gets better. Figure 4.3e show the error frequency response for approximate balanced-truncated model reduction for $N = 0, 1, 2$ and the balanced-truncated model reduction. The error is high at low frequencies then decreases at higher frequencies. Figure 4.3f shows the error frequency response for $N = 3, 4, 5$ and balancing. The error is very small for all three values of N. Figure 4.3g shows the error frequency response for $N \geq 6$ and balancing, where almost there is no difference between them.

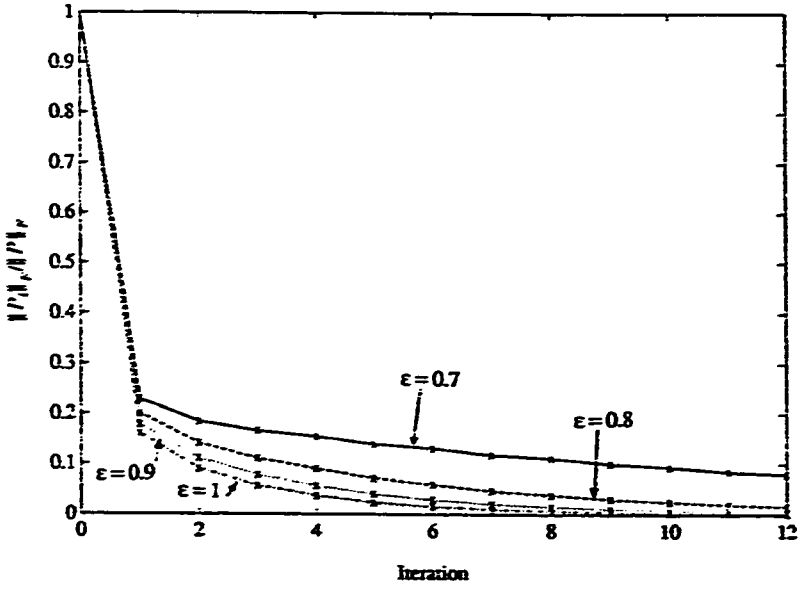

Fig. 4.3a : The normalized Frobenius norm of P_i.

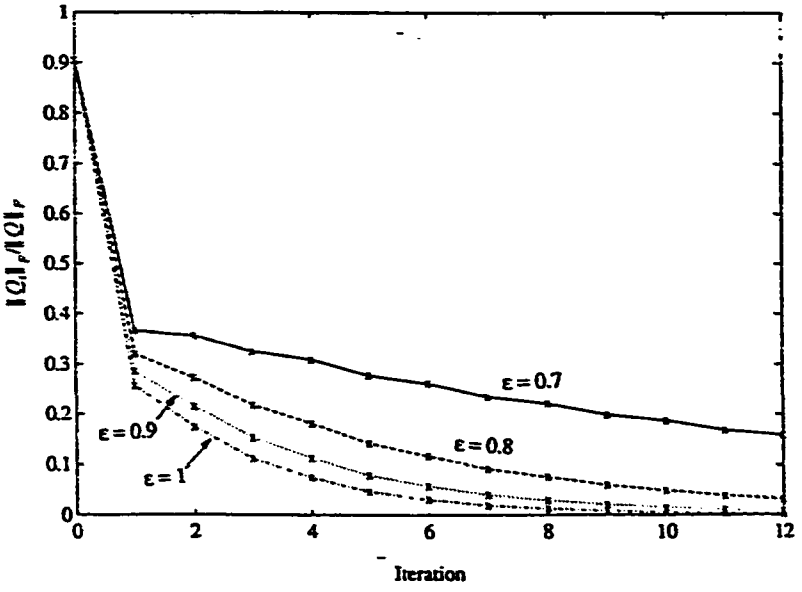

Fig. 4.3b : The normalized Frobenius norm of Q_i.

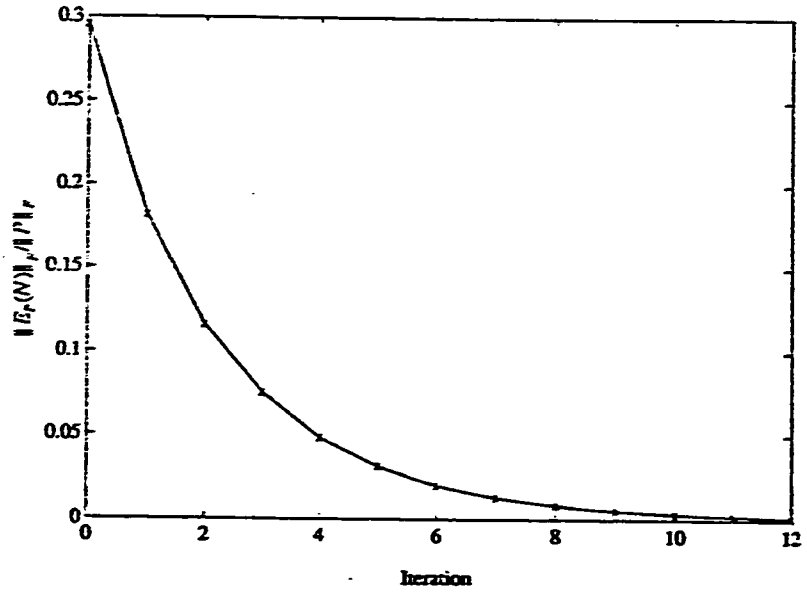

Fig. 4.3c : The normalized Frobenius norm of $E_P(N)$.

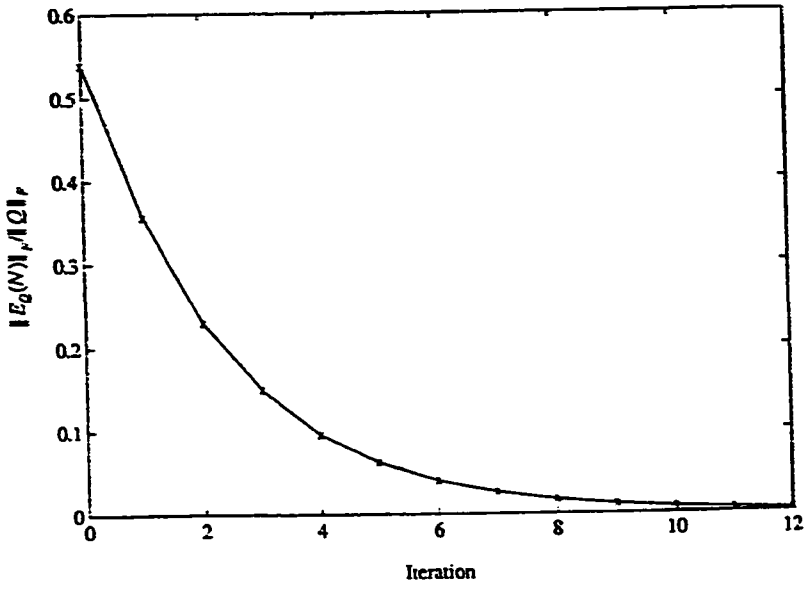

Fig. 4.3d : The normalized Frobenius norm of $E_Q(N)$.

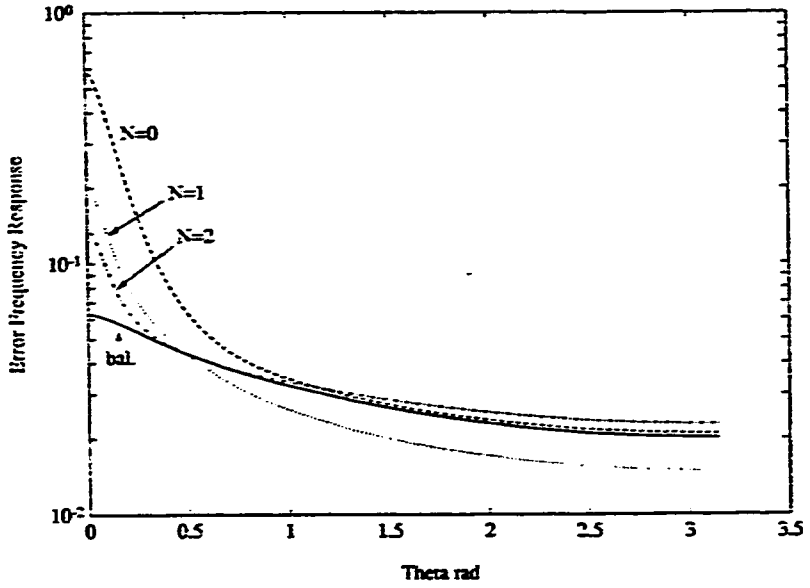

Fig. 4.3e : Error frequency response for N=0,1,2 and balancing.

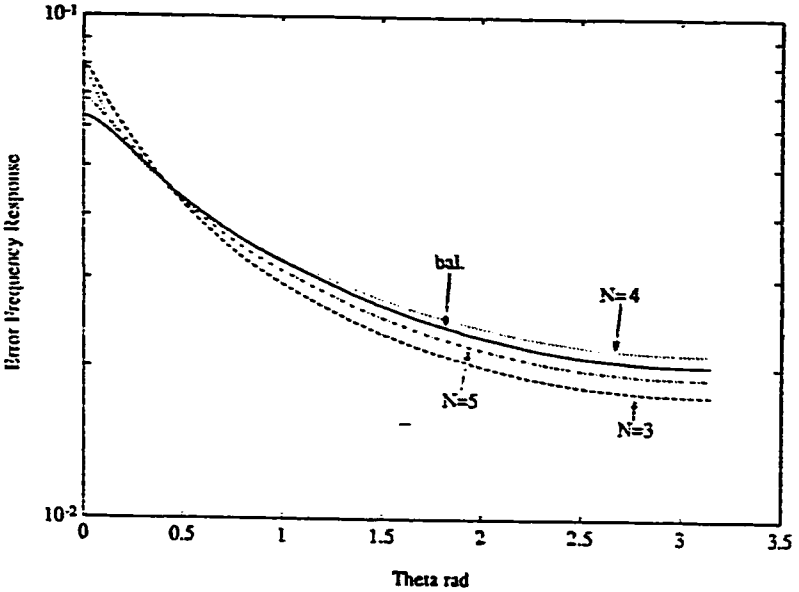

Fig. 4.3f : Error frequency response for N=3,4,5 and balancing.

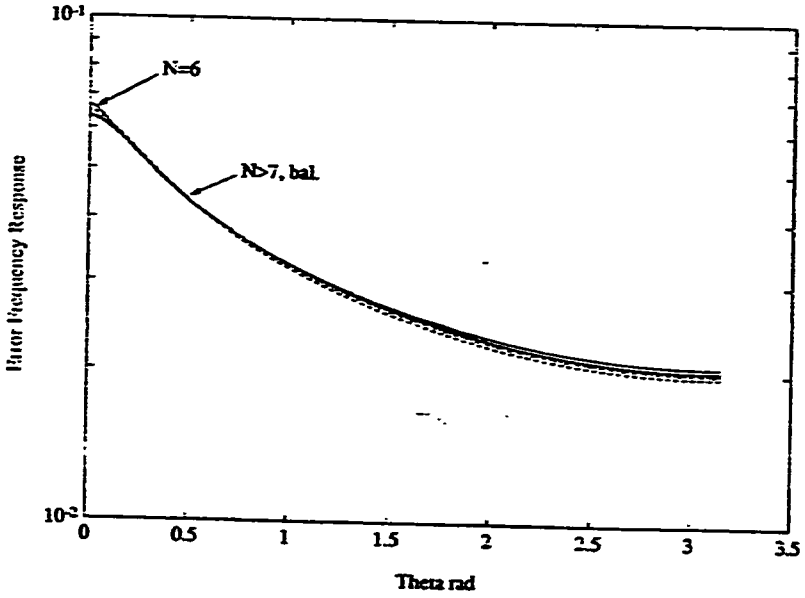

Fig. 4.3g : Error frequency response for $N \geq 6$ and balancing.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In this thesis, there are three main parts. The first one is producing a reduced order model with poles clustered in the same region as the full order model. This method can, also, be used for reduction of unstable systems, in this case the region should be chosen in both left and right hand planes or in the right hand plane if the system is antistable to make sure the poles are clustered inside it.

In the second part, the balanced model reduction is modified using bilinear transformation to make the frequency response of the error between the full order model and the reduced order model low pass instead of being high pass. The reduced order model is generically stable. This method is developed for both continuous-time and discrete-time systems.

Finally, approximate balanced representation and approximate balanced-truncation model reduction have been developed. Based on the notation of ε-coupling, the approximate controllability and observability gramains have been defined as power series expan-

sions in the coupling parameter ε. An iterative algorithm for computing them is developed and proved. Also, conditions were given for the convergence of the approximate gramians to the actual ones and bounds on the norms of the approximation errors are driven.

5.2 Recommendations

* Prove that the generalized balanced model reduction produces always a reduced order model with poles clustered in the same region as the full order model for any second order regions.

* Find a mathematical proof for the stability and minimality of reduced order models produced by generalized balanced model reduction.

* Find the bound on the peak value of the error frequency response of the generalized balanced model reduction.

* Find the bound on the peak value of the error frequency response of the balanced model reduction using bilinear transformation that was developed in chapter 3.

* Simplify the condition (4.36)

* Generalize lemma 4.3 to include necessary and sufficient conditions.

NOMENCLATURE

$R^{n\times m}, C^{n\times m}$	space of $n \times m$ real and complex matrices.
A', A^*	transpose and complex conjugate transposeof the matrix A.
$\sigma_i(A)$, $\bar{\sigma}(A)$	i-th and the largest singular value of A.
$diag(a_1, a_2, \ldots, a_n)$	diagonal matrix whose (i,i) element is a_i.
$\lambda(A)$, $\lambda_i(A)$	eigenvalue and i-th eigenvalue.
$\sigma(A)$	spectrum of A.
$\rho(A)$, $tr(A)$	spectral radius and trace of A.
P, Q	controllability and observability gramians.
Σ	controllability and observability gramians for balanced systems.
S	poles clustering region.
T	similarity transformation.
$\|X(s)\|_\infty$	infinity norm of $X(s)$.
$\|P\|_F$	Forbenius norm P.

$\|P\|$	Euclidean norm of P.
ε	coupling parameter.
$\otimes$	Kronecker product.

REFERENCES

1. Ahn, S. M.,"Eigenvalues of a matrix inside an ellipse and Chebyshev filter",*IEEE Trans. on Automat. Contr.*,AC-24,pp.986-987,1979.

2. Al-Saggaf, U. M.,"Approximately Balanced-Truncation Model Reduction Of Weakly Coupled Systems",*International Journal of Control*,(Accepted).

3. Al-Saggaf, U. M., "Model reduction for discrete unstable systems based on generalized normal representations", *International Journal of Control*, Vol 55, pp 431-443,1992.

4. Al-Saggaf, U. M., "On Model Reduction and Control of discrete-time Systems", Ph.D. dissertation, Information Systems Laboratory, Dept. of Electrical Engineering, Stanford University, June 1986.

5. Al-Saggaf, U. M., "Subsystem Interconnection and Near Optimum Control of Discrete Balanced Systems", *IEEE Trans. Automat. Control* (In Press).

6. Al-Saggaf, U. M. and G.F. Franklin, "An Error Bound for a Discrete Reduced Order Model of a Linear Multivariable System", *IEEE Trans. on Automatic Control*, Vol. AC-32, No. 9, pp. 815-819, Sept., 1987.

7. Al-Saggaf, U. M. and G.F. Franklin, "Model Reduction via Balanced Realizations, an Extension and Frequency Weighting Techniques", *IEEE Trans. on Automatic Control*, Vol. AC-33, No. 7, pp. 687-692, July 1988.

8. Bahnasawi, A. A., A. S. Al-Fuhaid and M. S. Mahmoud,"Decetralised and hierarchical control of interconnected uncertain systems",*IEE Proc. Pt D*,173,pp.311-321,1990

9. Belloch, P. A., D. L. Mingori and J. D. Wei, "Perturbation Analysis of Internal Balancing for Lightly Damped Mechanical Systems with Gyroscopic and Circulatory Forcess", *AIAA Journal of Guidance and Control* 10(4), pp. 406-410 (1987).

10. Bettayeb, M., "Approximation of linear systems: New Approaches based on Singular value decomposition", Ph.D. Thesis, University of Southern California, Los Angeles, August 1981.

11. Bettayeb, M., L.M. Silverman and M.G. Safonov, "Optimal approximation of continuous time systems, "Proc. *19th I.E.E.E. CDC*, Albuquerque, New Mexico, December 1980.

12 Bettayeb, M. and S. Djennoune,"New results on the interconnection between subsystems of a balanced representation," Accepted in Rairo, AP II, 1989.

13. Bettayeb, M. and S. Djennoune,"Structure of singularly perturbed systems and balancing", *Accepted, IEE Proc. Pt.D*, 1991.

14. Chen, C. F. and L. S. Shieh,," A novel approach to linear model simplification," *Int. J. Contr.*, vol. 8,pp 561-570,1968

15. Davison, E. J. and N. Ramesh,"A note on the eigenvalue of a real matrix",*IEEE Trans. Automat. Contr.*,pp.252-253,1970.

16. Enns, D. F., "Model Reduction for Control System Design," Ph.D. dissertation, Dept. of Aeronautics and Astronautics, Stanford University, June 1984.

17. Fernando, K. V. and H. Nicholson, "On the structure of balanced and other principal representations of SISO systems, *IEEE Trans. Automat. Contr.*, AC-28, PP. 228-231 (1983).

18. Fernando, K. V. and H. Nicholson, "Reciproical transformations in balanced model-order reduction *proceedings of the Institution of Electrical Engineers, Pt D*, **130, pp. 359-362 (1983).**

19. Fernando, K. V. and H. Nicholson, "Singular perturbation model reduction of balanced systems", *IEEE Trans. Automat. Contr.* AC-27, pp. 466-468 (1982).

20. Fernando, K. V. and H. Nicholson, "Singular perturbational model reduction in the frequency domain", *IEEE Trans. Automat. Contr.* AE-27, PP. 969-970 (1982).

21. Fu, X. and L. Fu,"Optimal Control System Design of a Nuclear Reactor By Generalized Spectral Factorization",Int.J.Contro.,Vol 47,pp.1479-1487,1988.

22. Genesio, R., and M. Milanese,"A note on the derivation and use of reduced order models," *IEEE Trans. Automat. Contr.*, Vol. AC-21, pp. 118-122, 1976.

23. Glover, K.,"All optimal Hankel-norm approximations of Linear Multivariable systems and their L^{∞}-error bounds", *Int. J. Control*, Vol. 39, No. 6, pp. 1115-1193, 1984.

24. Gopal, M. and S. I. Mehta,"On the selection of the eigenvalues to be retained in the order models,"*IEEE Trans. Automat. Contr.* AC-27,pp. 688-690,(1982).

25. Graham, A.,"Kronecker Products and Matrix Computations",2nd edition,Johns Hopkins University press,1981.

26. Gregory, C. Z., "Reduction of Large Flexible Spacecraft Models Using Internal Balancing Theory", *AIAA Journal of Guidance and Control* 7, pp. 725-732 (1984).

27. Gutman, S.,"Matrix root clustering in algebriac regions",*Int.J.Contr.*,vol 39,pp.773-778,1984.

28. Gutman, S.,"Root Clustering of a Real Matrix In an Algebriac Region",*Int.J.Contr.*,vol 29,pp.871-881,1979.

29. Gutman, S.,"Roots Clustering of a Complex Matrix in an Algebraic Region",*IEEE Trans. on Automat. Contr.*,AC-24,pp.647-650,1979.

30. Han, K. W.,"Reducing the Effects of Model Reduction Due to Parameter Variations",in "Control and Dynamic Systems", vol. 31 (C. T. Leondes. ed.),Academic Press, San Diego,1989.

31. Jamshidi, M.,"Large-Scale Systems Modeling and Control", North-Holland, 1983.

32. Jonckheere, E. A., "Principal Component Analysis of Flexible Systems-Open Loop Case", *IEEE Trans. Automat. Contr.* AC-29, pp. 1095-1097 (1984).

33. Jonckheere, E. A. and L. M. Silverman, "Singular value analysis of Deformable Systems", *Circuits, Systems Signal Process*, Vol. 1, Birkhauser, Boston (1982).

34. Jonckheere, E. A. and P. Opendenacker, "Singular value analysis, Balancing and Model Reduction of Large Space Structures", *Proceedings of American Control Conference*, (1984).

35. Jury, E. I.,"Inners and Stability of Dynamic Systems",2nd ed.,Robert E. Krieger,Malabar,Florida,1984.

36. Kenney, C. and G. Hewer, "Necessary and Sufficient Conditions for balancing unstable systems", *IEEE Trans. Automat. Contr.* AC-32, pp. 157-160 (1987).

37. Kokotovic, P. V.,"Feedback design od large-scale linear systems",In J. B. Cruz,Jr.,ed.,*Feedback Systems*,pp.99-137.McGraw-Hill,New York,1972.

38. Kokotovic, P. V.,W. R. Perkins,B. J. Cruz and G. D'Ans,"ε-Coupling Method for near-optimum design of large-scale linear systems",*Proc. IEE*,116,1969.

39. Laub, A. J., M.T. Heath, C.C. Paige and R.C. Ward, "Computation of System balancing Transformations and other Applications of simultaneous Diagonalization Algorithms",*IEEE Transactions on Automatic Control*, Vol. AC-32, No. 2, pp. 115-121, 1987.

40. Lin, C. and T. Chiu,"Model Reduction via Frequency Weighted Balanced Realization",Proc. of Amer. Cont. Conf.,pp 2069-2070,1990.

41. Liu, Y. and B. D. O. Anderson,"Sigularly Perturbational Approximation of Balanced Systems,"*Int.J.Contr.*,Vol 50,pp. 1379-1405,1989.

42. Mahamoud, M. S. and M. G. Singh,"Large Scale Systems Modelling",*Inter. Series on Systems and Control*,Vol 3,Pergamon press,1985.

43. Meier, L., III and D.G. Luenberger, "Approximation of linear Constant Systems," *IEEE Trans. Automat. Contr.*, Vol. AC-12, pp. 585-588, 1967.

44. Moore, B. C., "Principal components analysis in linear systems: Controllability, observability and model reduction", *IEEE Transactions on Automatic control*, Vol. AC-26, pp. 17-32, 1981.

45. Mullis, C. T. and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters", *IEEE Trans. Circuits Syst.* CAS-23, pp. 551-561 (1976).

46. Pernebo, L. and L. M. Silverman, "Model Reduction via balanced state space representations," *IEEE Trans. Automat. Contr.*, vol. AC-27, No. 2, Apr. 1982.

47. Safanov, M. G. anf R. Y. Chiang,"CACSD Using The State-Space L_∞ Theory - A Design Example",*Proc. IEEE Conf. on CACSD*,Washington D. C.,1986,also,*IEEE Trans. on Automat. Contr.*,AC-33,No.5,pp. 477-479,1988

48. Safonov, M. G. and R. Y. Chiang, "A Schur method for balanced-truncation model reduction", *IEEE Trans. Automat. Contr.*, AC-34, pp. 729-733 (1989).

49. Shahruz, S. M. and S. Behtash,"Balanced Realizations of Singularly Perturbed Syatems",*Int.J.Contr.*,Vol 49,pp.207-217,1989.

50. Shamash, Y., "Model reduction using the Routh stability criterion and Pade' approximations technique",*Int.J.Contr.* Vol 21,No. 3, pp. 475-484,(1975).

51. Shokoohi, S., L. M. Silverman, and P. M. Van Dooren, "Linear Time-Variable Systems: Balancing and Model Reduction", *IEEE Transactions on Automatic Control,* Vol. AC-28, Aug. 1983, pp. 810-822.

52. Shokoohi, S. and L. M. Silverman, "Model Reduction of discrete time-variable systems via balancing", proc. of 20th IEEE Conf. on Dec. & Control, pp. 676-680 (1981).

53. Silverman, L. M. and M. Bettayeb, "Optimal approximation of linear Systems", in *JACC, Vol. 2, p. FA 8-A,* August 1980.

54. Sreeram, V. and P. Agathoklis,"Model reduction using balanced realization with improved low frequency behaviour",*Systems and Control Letters*,12,pp.33-38,1989

55. Therapos, C. P., "Balancing-Transformations for unstable non-minimal linear systems", *IEEE Trans. Automat. Contr.* AC-34, pp. 455-457., 1989.

56. Troullinos, G. and F. J. Dorsey,"Application of balanced realizations to power system equivalents",*IEEE Trans. Automat. Contro.*,AC-30,pp.414-416,1985

57. Verriest, E. I. and T. Kailath, "On Generalized Balanced Realizations", *IEEE Transactions on Automatic Control,* Vol. AC-28, Aug. 1983, PP. 833-844.

58. Wilson, D. A., "Model reduction for multivariable systems," *Int. J. Contr.*, vol. 20, pp. 57-64, 1974.

59. Wilson, D. A., "Optimum Solution of model reduction problem," *Proc. IEE,* vol. 117, June 1970.

60. Jonckheere, E. and R. Li, "Generalization of optimal Hankel-norm and balanced model reduction by bilinear mapping" *Int. J. Contr.* 45(5), pp. 1751-1769 (1987).

61. Gahinet, P., A. J. Aaub, C. S. Kenney and G Hewer,"Sensitivity of the Stable Discrete-Time Lyapunov Equation," *IEEE Trans. On Automatic Control,* Vol AC-35, Nov. 1990,pp 1209-1217.

62. O'Neil, Peter V.,"Advanced Engineering Mathematics," Wadsworth Publishing Co., Belmont, California,1987.

63. Chen, Carson,"Active Filter Design," Hayden Book Co. Inc., New Jersey, 1982.

64. Al-Saggaf, U. M. and M. Bettayeb,"Techniques in Optimized Reduction for High Dimentional Systems." Invited chapter in International Academic Press Series,"Advances in Control and Dynamic Systems", Vol 49-52, Spring 1992.