2. Mathematical Models of Systems (cont.)

Example 3 Solution of a differential equation

Obtain the the response of the system represented by the differential equation:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = 2r(t)$$

Where the initial conditions are $y(0)=1$, $\frac{dy}{dt}(0)=0$, and $r(t)=1,\ t \geq 0$.

Solution

The Laplace transform yields

$$[s^2Y(s) - sy(0)] + 4[sY(s) - y(0)] + 3Y(s) = 2R(s)$$

$$[s^2Y(s) - s] + 4[sY(s) - 1] + 3Y(s) = 2\frac{1}{s}$$

$$Y(s) = \frac{s+4}{(s^2+4s+3)} + \frac{2}{s(s^2+4s+3)}$$

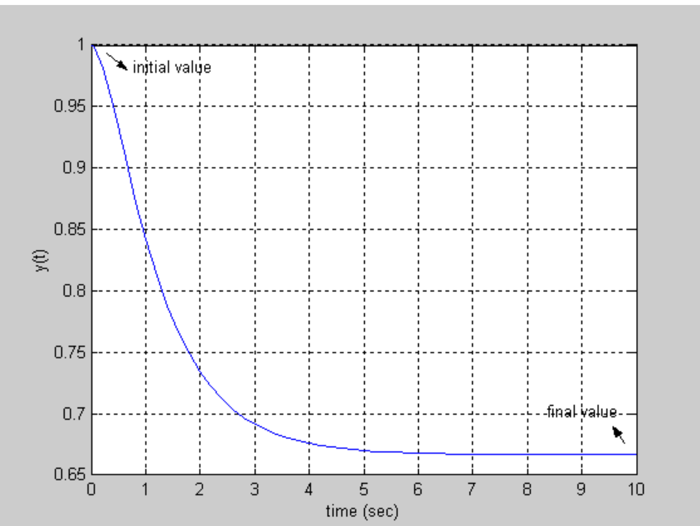
Where $(s^2+4s+3) = (s+1)(s+3) = 0$ is the characteristic equation. Then the partial fraction expansion yields

$$Y(s) = [\frac{\frac{3}{2}}{(s+1)} + \frac{-\frac{1}{2}}{(s+3)}] + [\frac{-1}{(s+1)} + \frac{\frac{1}{3}}{(s+3)}] + \frac{\frac{2}{3}}{s}$$

Hence the response is $y(t) = [\frac{3}{2}\in^{-t} -- \frac{1}{2}\in^{-3t}] + [-\in^{-t} + \frac{1}{3}\in^{-3t}] + \frac{2}{3}$

$y(t) = \frac{1}{2}\in^{-t} - \frac{1}{6}\in^{-3t} + \frac{2}{3}$ and the steady-state response is $\lim_{t\to\infty} y(t) = \frac{2}{3}$

The response is shown in the figure below.



Drill Problem [to be submitted]

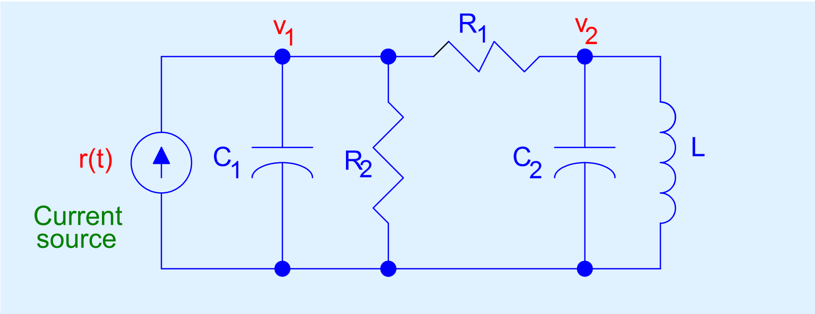
Simulate the above system using simulink and plot the response $y(t)$ for the following cases:

1. $y(0) = 0$, $\frac{dy}{dt}(0) = 0$, and $r(t) = 1$, $t \geq 0$.
2. $y(0) = 1$, $\frac{dy}{dt}(0) = 0$, and $r(t) = 0$, $t \geq 0$.
3. $y(0) = 1$, $\frac{dy}{dt}(0) = 0$, and $r(t) = 1$, $t \geq 0$.

Comment on the results and show your simulink model.

Example 4

Find the transfer functions $\frac{V_1(s)}{R(s)}$ and $\frac{V_2(s)}{R(s)}$ for the given circuit



Solution

The transfer functions are obtained by writing the node equations, yielding

$$C_1 s V_1(s) + \frac{V_1(s)}{R_2} + \frac{V_1(s) - V_2(s)}{R_1} = R(s) \quad ;$$

$$C_2 s V_2(s) + \frac{V_2(s)}{sL} + \frac{V_2(s) - V_1(s)}{R_1} = 0 \quad ;$$

or, in matrix form, we have

$$\begin{bmatrix} C_1 s + \frac{1}{R_2} + \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R1} & sC_2 + \frac{1}{sL} + \frac{1}{R_1} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

$$\frac{V_1(s)}{R(s)} = \frac{\left(sC_2 + \frac{1}{sL} + \frac{1}{R_1}\right)}{\left(C_1 s + \frac{1}{R_2} + \frac{1}{R_1}\right)\left(sC_2 + \frac{1}{sL} + \frac{1}{R_1}\right) - \frac{1}{R_1^2}}$$

$$\frac{V_2(s)}{R(s)} = \frac{\left(\frac{1}{R_1}\right)}{\left(C_1 s + \frac{1}{R_2} + \frac{1}{R_1}\right)\left(sC_2 + \frac{1}{sL} + \frac{1}{R_1}\right) - \frac{1}{R_1^2}}$$