8. FREQUENCY RESPONSE METHODS (CONT.)

LOGARITHMIC PLOTS [BODE PLOTS]

The use of logarithmic plots, often called Bode plots, simplifies the determination of the graphical portrayal of the frequency response. The transfer function in the frequency domain is

$$G(j\omega) = |G((j\omega|\, e^{j\phi(\omega)}$$

The Bode plots consist of two plots:

- ✶ Magnitude in decibels (dB) [i.e. $20\log_{10}(|G((j\omega|)]$ versus the frequency $\log_{10}(\omega)$.
- ✶ Phase angle $[\phi(\omega)]$ versus the frequency $\log_{10}(\omega)$.

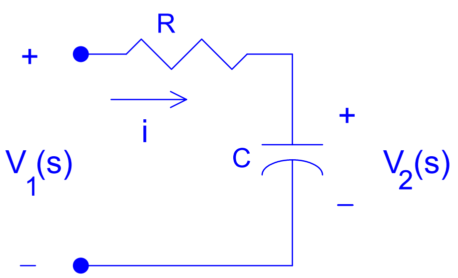
- As **logarithmic plots**, often called **Bode plots**, representing separately the magnitude and phase, as a function of frequency. The magnitude curve can be plotted in decibels (**dB**) vs. log ω, where $dB = 20\log|G(j\omega)|$. The phase curve is plotted as phase angle vs. log ω .

An example of logarithmic plots is given next:

Example 1 Bode plots of an RC filter

A simple RC filter us shown. The transfer function of this filter is

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs+1}$$



The sinusoidal steady-state transfer function is

$$G(j\omega) = \frac{1}{j\omega(RC)+1} = \frac{1}{j\omega\tau+1} \quad ; \quad \tau = RC$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{1}{\sqrt{1+(\omega\tau)^2}} = \quad -10\log_{10}[1+(\omega\tau)^2]$$ or

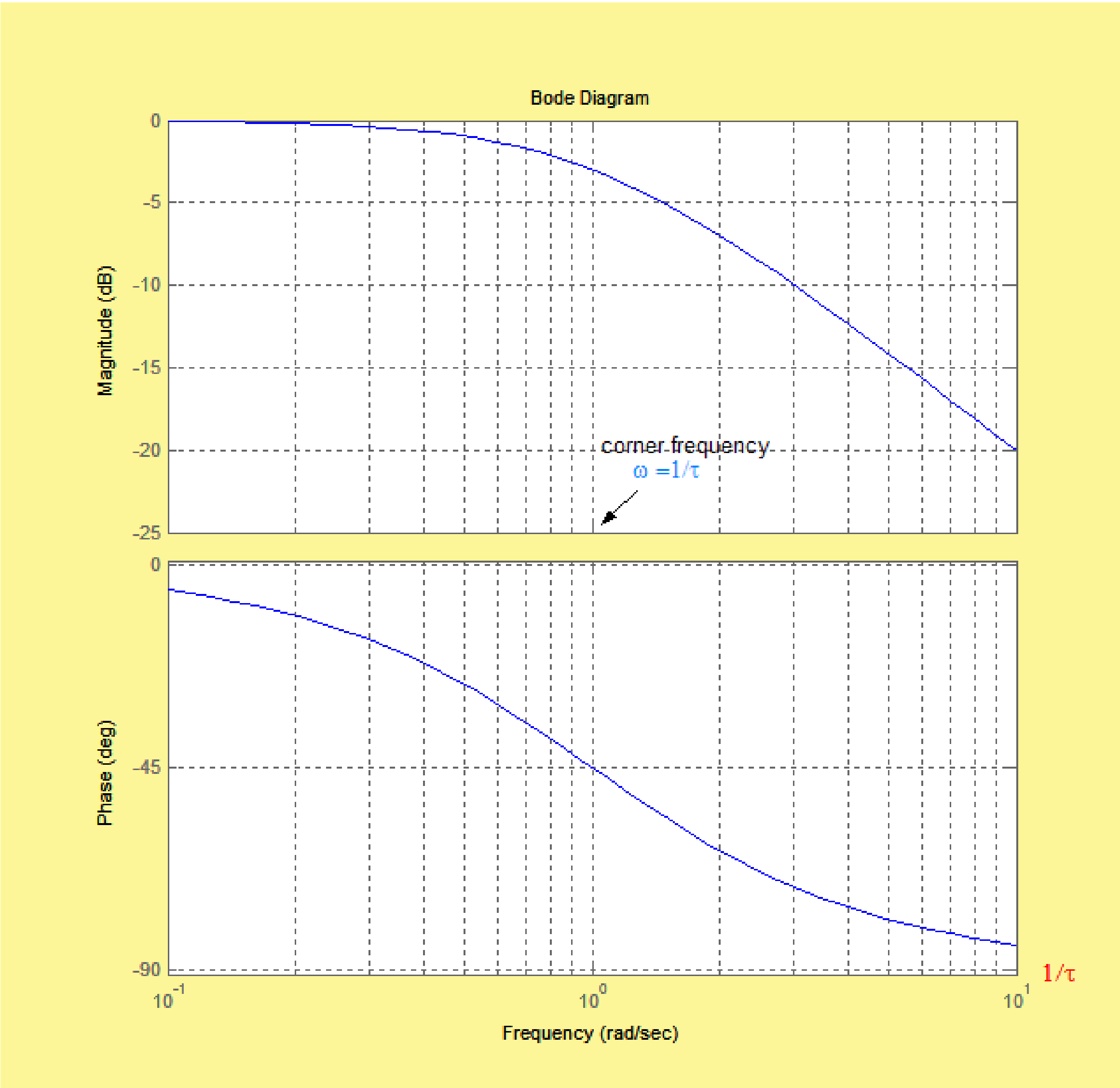
$$\phi(\omega) = \angle -\tan^{-1}(\omega\tau)$$

The phase angle and the magnitude are readily calculated at the frequencies $\omega = 0, \omega = \frac{1}{\tau}$, and ∞. The values are given in the table.

ω	0	$\frac{1}{\tau}$	$+\infty$
$\lvert G(\omega)\rvert$	1	$\frac{1}{\sqrt{2}}$	0
dB	0	-3.01	$-\infty$
$\phi(\omega)$	0°	-45°	-90°

The frequency $\omega = \frac{1}{\tau}$ is often called the **break frequency** or **corner frequency**

The magnitude plot and the phase plot are shown.



The primary advantage of the logarithmic plot is the conversion of multiplicative factors such as $(j\omega\tau+1)$ into additive factors $20\log_{10}(j\omega\tau+1)$ by virtue of the definition of logarithmic gain. This can be readily ascertained by considering a generalized transfer function as

$$G(j\omega)=K_b\frac{\prod_{i=1}^{Q}(1+j\omega\tau_i)}{(j\omega)^N\prod_{m=1}^{M}(1+j\omega\tau_m)\prod_{k=1}^{R}[1+j2\zeta_k\frac{\omega}{\omega_{n_k}}+\left(j\frac{\omega}{\omega_{n_k}}\right)^2]}$$

Obtaining the polar plot of such a function would be a formidable task indeed. However, the logarithmic magnitude of $G(j\omega)$ is:

$$\begin{aligned}20\log_{10}|G(j\omega)| &= 20\log_{10}K_b+20\sum_{i=1}^{Q}\log_{10}|1+j\omega\tau_i|-20N\log_{10}|j\omega|\\ &\quad -20\sum_{m=1}^{M}\log_{10}|1+j\omega\tau_m|-20\sum_{k=1}^{R}\log_{10}\left|1+j2\zeta_k\frac{\omega}{\omega_{n_k}}+\left(j\frac{\omega}{\omega_{n_k}}\right)^2\right|\end{aligned}$$

and the Bode plot can be obtained by adding the plot due to each individual factor.

Furthermore, the separate phase angle plot is obtained as

$$\phi(\omega)=+\sum_{i=1}^{Q}\tan^{-1}\omega\tau_i-N(90^\circ)-\sum_{m=1}^{M}\tan^{-1}\omega\tau_m-\sum_{k=1}^{R}\tan^{-1}\left(\frac{2\zeta_k\frac{\omega}{\omega_{n_k}}}{1-\left(\frac{\omega}{\omega_{n_k}}\right)^2}\right)$$

which is simply the summation of the phase angles due to each individual factor of the transfer function.

The four different kinds of factor that may occur in a transfer function are as follows:

- Constant gain K
- Poles (or zeros) at the origin $(j\omega)$
- Poles (or zeros) on the real axis $(j\omega\tau+1)$
- Complex poles (or zeros) $\left[1+j2\zeta\frac{\omega}{\omega_n}+\left(j\frac{\omega}{\omega_n}\right)^2\right]$

We can determine the logarithmic magnitude plot and phase angle for these four factors and then utilize them to obtain a bode diagram for any general form of a transfer function. Typically the curves for each factor are obtained and then added together graphically to obtain the curves for the complete transfer function. Furthermore this procedure can be simplified by using the asymptotic approximations to these curves and obtaining the actual curves only at specific important frequencies.

Asymptotic Approximations:

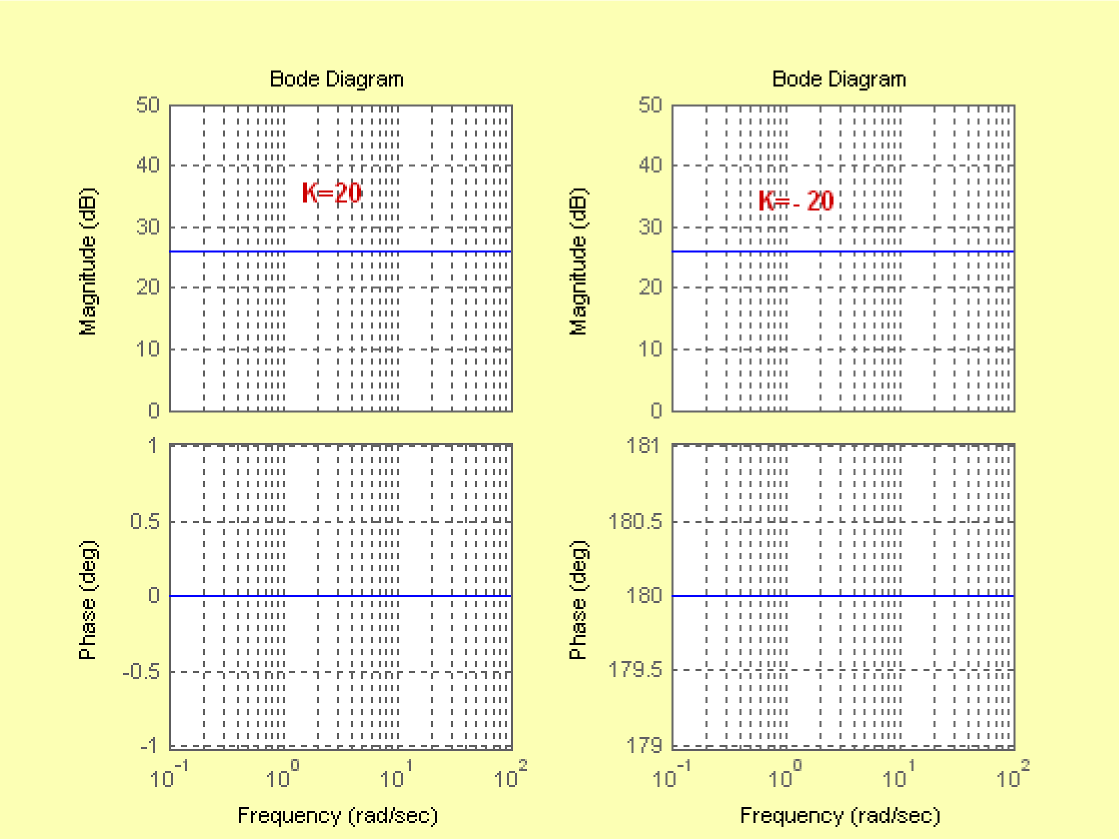
Let us now show how to approximate the frequency response of each of the above four kinds of factors by straight line approximations. Later, we show how to combine these responses to sketch the frequency response of more complicated function.

Constant Gain K_b

$$\boxed{20\log_{10} K_b = \text{constant in dB}}$$

$$\boxed{\phi(\omega) = 0}$$

The gain curve is simply a horizontal line on the bode plot. If the gain is a negative value, $-K_b$, the logarithmic gain remains $20\log_{10} K_b$, the negative sign is accounted for by the phase angle, -180°. The bode plots for $K = 20$ and $K = -20$ are shown .



Poles (or Zeros) at the origin $j\omega$

Pole at the origin

$$\boxed{20\log_{10}\left|\frac{1}{j\omega}\right| = -20\log_{10}\omega \ \text{ dB}}$$

$$\boxed{\phi(\omega) = -90^\circ}$$

- The slope of the magnitude curve is $-20\,\text{dB/decade}$.

Multiple Pole at the Origin

$$20\log_{10}\left|\frac{1}{j\omega^N}\right| = -20N\log_{10}\omega \;\; \text{dB}$$

$$\phi(\omega) = N(-90^\circ)$$

- The slope of the magnitude curve is $-20N$ dB/decade.

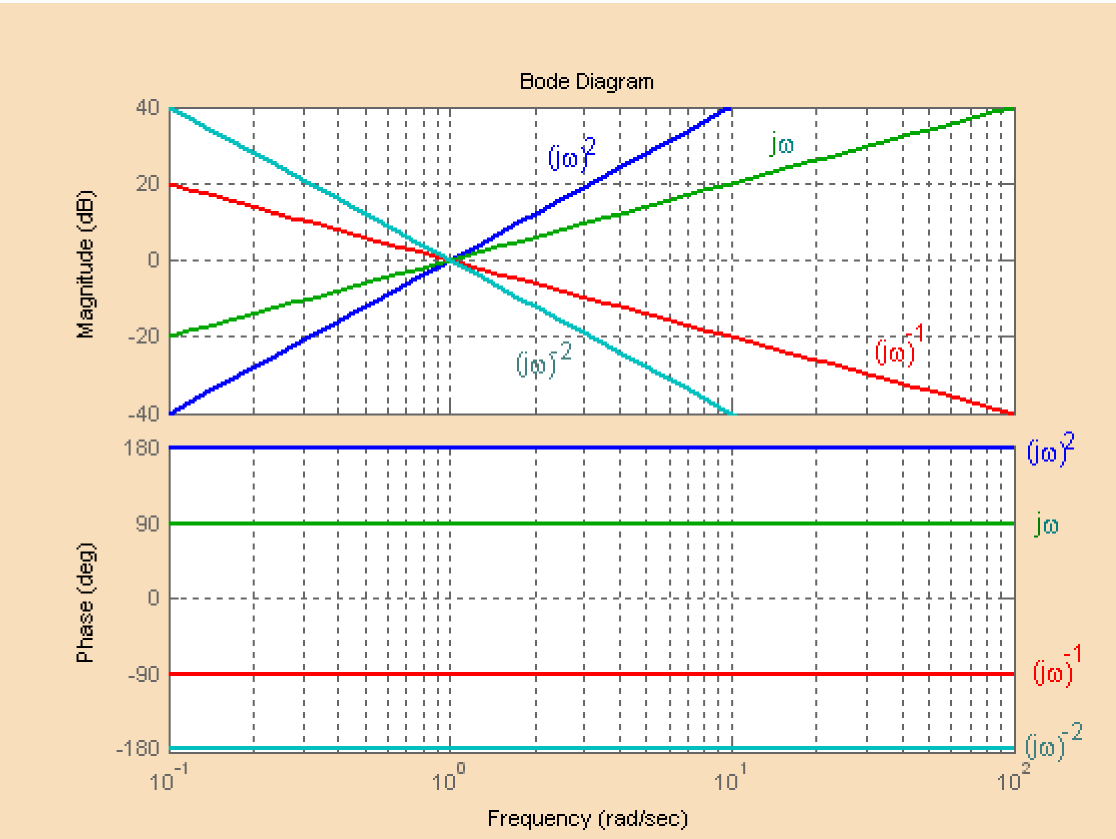
Zero at the Origin

$$20\log_{10}|j\omega| = +20\log_{10}\omega \;\; \text{dB}$$

$$\phi(\omega) = +90^\circ$$

- The slope of the magnitude curve is $+20$ dB/decade.

The bode plots of $(j\omega)^{\pm N}$ are shown for $N = 1$ and $N = 2$.



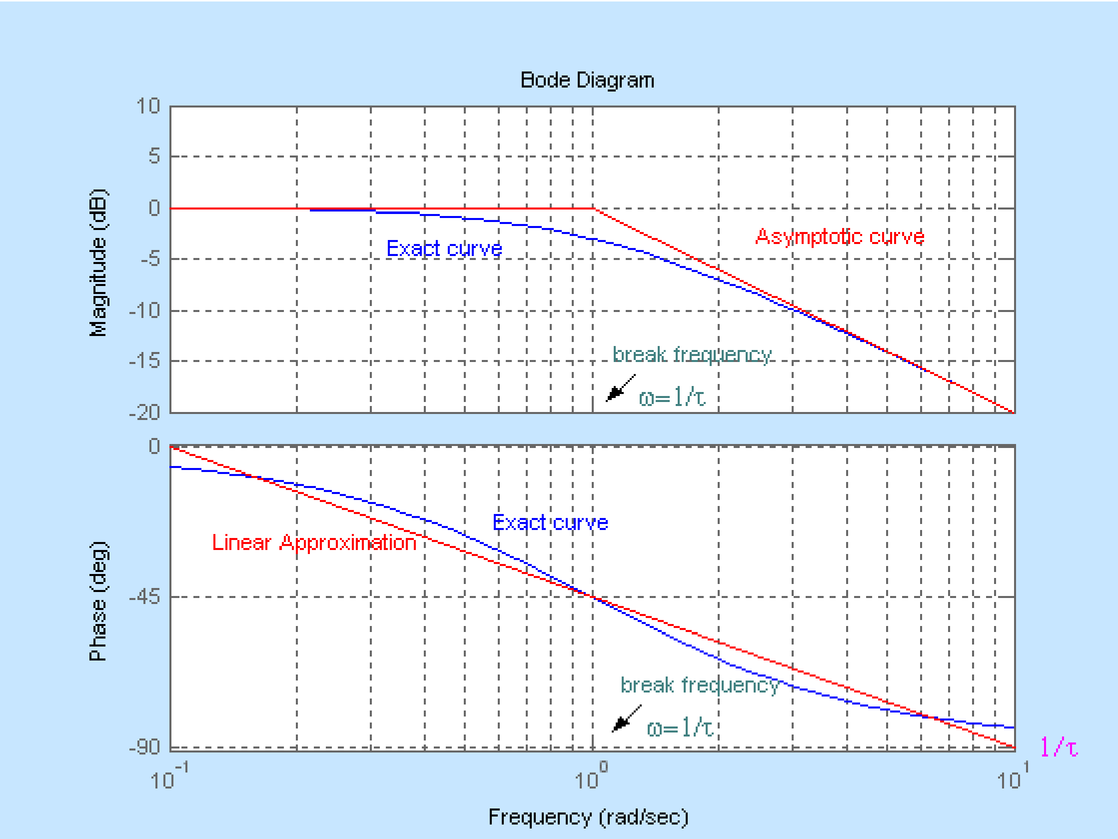
Poles (or Zeros) on the real axis $(1+j\omega\tau)$

Pole on the real axis

$$20\log_{10}\left|\frac{1}{1+j\omega\tau}\right| = -10\log_{10}(1+\omega^2\tau^2)\ \text{dB}$$

$$\phi(\omega) = -\tan^{-1}\omega\tau$$

- The asymptotic curve for $\omega << \frac{1}{\tau}$ is $20\log_{10}1 = 0$ dB
- The asymptotic curve for $\omega >> \frac{1}{\tau}$ is $-20\log_{10}\omega\tau$ dB, which has a slope of -20 dB/decade.
- The intersection of the two asymptotes occurs when $20\log_{10}1 = 0\,\text{dB} = -20\log_{10}\omega\tau$ or when $\omega = \frac{1}{\tau}$, the break frequency.
- The actual logarithmic gain when $\omega = \frac{1}{\tau}$ is -3 dB for this factor.
- The bode diagram of a pole factor $\frac{1}{1+j\omega\tau}$ is shown



Zero on the real axis

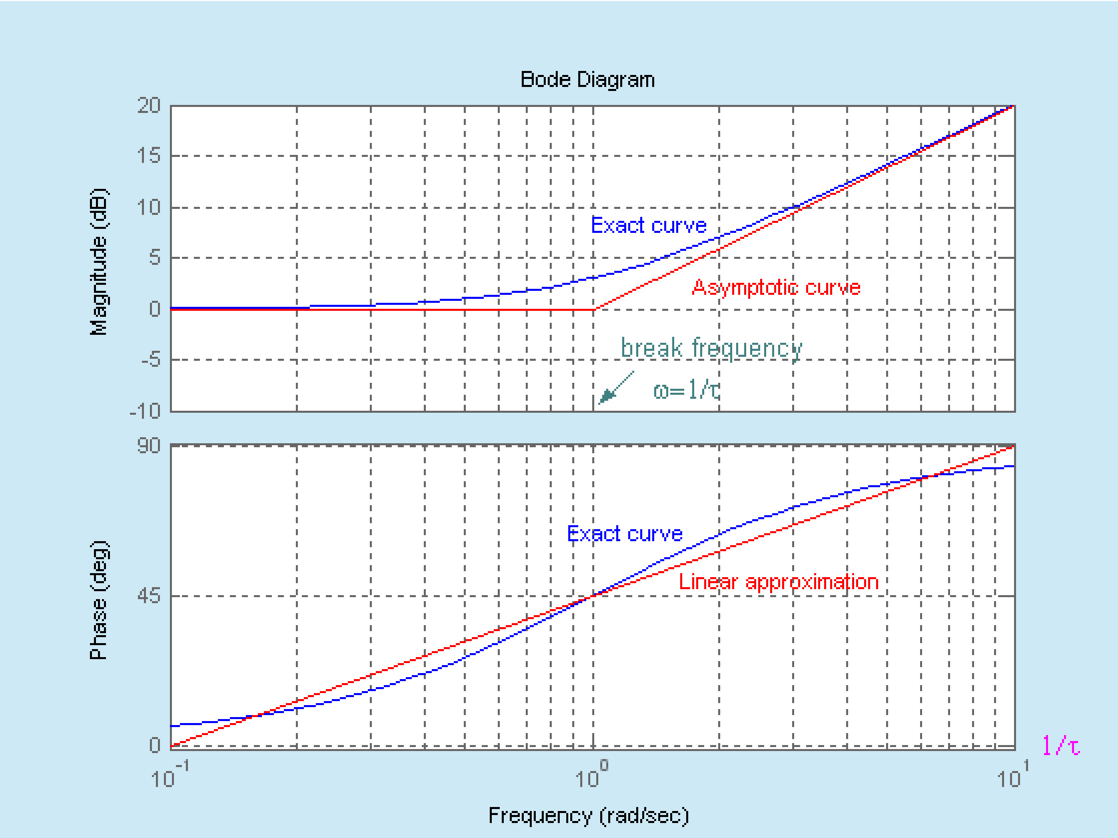
The bode diagram of a zero factor $(1+j\omega\tau)$ is obtained in the same manner as that of the pole. However, the slope is positive at $+20$ dB/decade

$$20\log_{10}|1+j\omega\tau| = +10\log_{10}(1+\omega^2\tau^2) \ \ \text{dB}$$

and the phase angle is

$$\phi(\omega) = +\tan^{-1}\omega\tau$$

- The bode diagram of a zero factor $(1+j\omega\tau)$ is shown



Complex Conjugate Poles (or Zeros) $\left(1+j2\zeta\frac{\omega}{\omega_n}+\left(j\frac{\omega}{\omega_n}\right)^2\right)$

The logarithmic magnitude for a pair of complex conjugate poles is

$$20\log_{10}\left|\frac{1}{\left(1+j2\zeta\frac{\omega}{\omega_n}+\left(j\frac{\omega}{\omega_n}\right)^2\right)}\right| = -10\log_{10}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right] \ \ \text{dB}$$

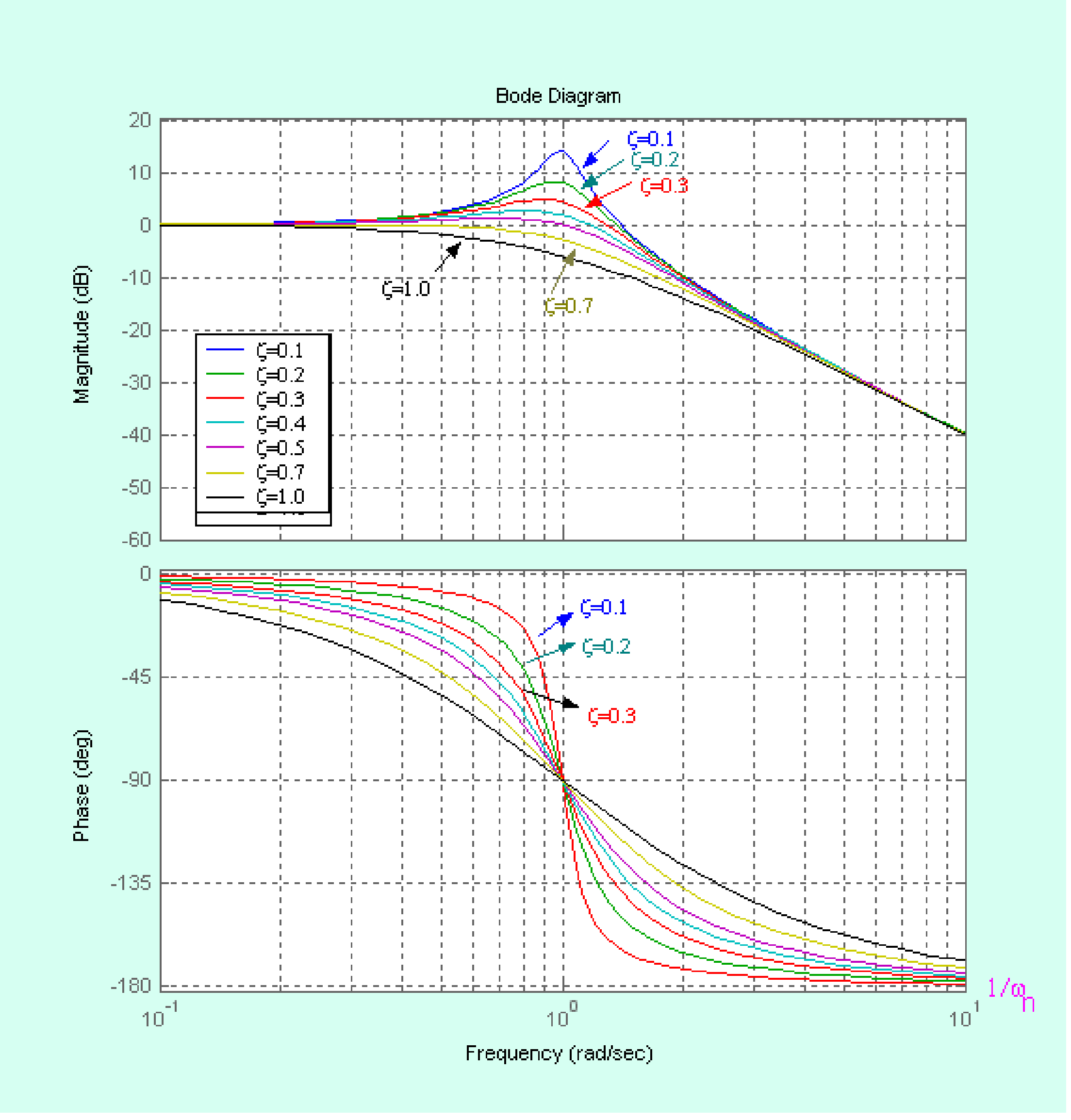
and the phase angle is

$$\phi(\omega) = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

- When $\frac{\omega}{\omega_n} << 1$, the magnitude is $-10\log_{10} 1 = 0$ dB, and the phase angle approaches 0°.
- When $\frac{\omega}{\omega_n} >> 1$, the magnitude is $-10\log_{10}\left(\frac{\omega}{\omega_n}\right)^4 = -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$ dB, which results in a curve with a slope of -40 dB/decade. The phase angle approaches -180°.
- The magnitude asymptotes meet at the $0-$dB line when $\frac{\omega}{\omega_n} = 1$.

Unlike the first-order frequency response approximation, the difference between the asymptotic approximation and the exact frequency response must be accounted for $\zeta < 0.707$.

The bode plots of a quadratic factor due to a pair of complex conjugate poles are shown



The maximum value of the frequency response, M_{p_ω}, occurs at the resonant frequency ω_r.

The resonant frequency is determined by taking the derivative of the magnitude of

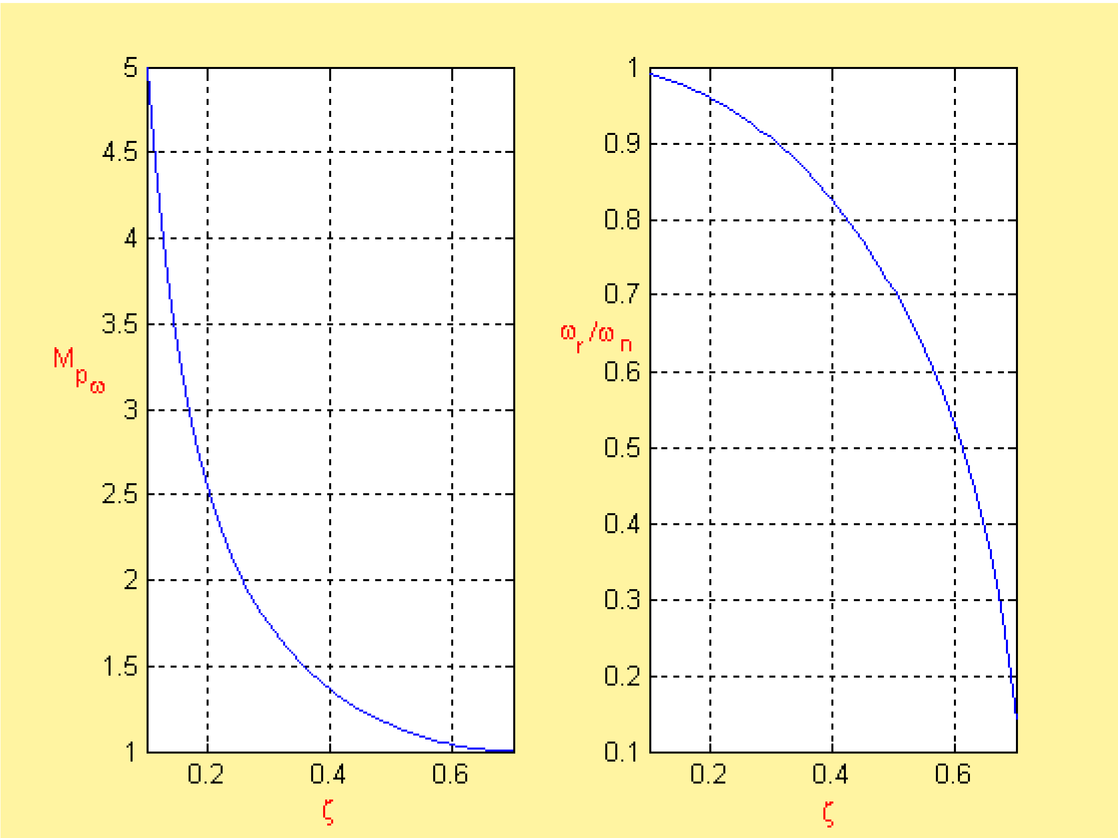
$$\frac{1}{\left(1+j2\zeta\frac{\omega}{\omega_n}-\left(\frac{\omega}{\omega_n}\right)^2\right)}$$

With respect to ω, and setting it equal to zero. The resonant frequency and the maximum value of the magnitude $|G(\omega)|$ are given by

$$\omega_r=\omega_n\sqrt{1-2\zeta^2} \qquad \zeta<0.707$$

$$M_{p_\omega}=|G(\omega_r|=\frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \zeta<0.707$$

The maximum value of the frequency response, M_{p_ω} , and the resonant frequency ω_r are shown as a function of the damping ratio ζ.



Complex conjugate zeros :

The bode plots of a pair of complex conjugate zeros $\left(1+j2\zeta\frac{\omega}{\omega_n}+\left(j\frac{\omega}{\omega_n}\right)^2\right)$ are obtained in the same manner as that of the complex conjugate poles. However, the slope is positive at $+40$ dB/decade.

The logarithmic magnitude for a pair of complex conjugate zeros is

$$20\log_{10}\left|\left(1+j2\zeta\frac{\omega}{\omega_n}+\left(j\frac{\omega}{\omega_n}\right)^2\right)\right| = +10\log_{10}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right] \text{ dB}$$

and the phase angle is

$$\phi(\omega)=\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

- When $\frac{\omega}{\omega_n} << 1$, the magnitude is $10\log_{10}1 = 0$ dB, and the phase angle approaches 0°.
- When $\frac{\omega}{\omega_n} >> 1$, the magnitude is $10\log_{10}\left(\frac{\omega}{\omega_n}\right)^4 = 40\log_{10}\left(\frac{\omega}{\omega_n}\right)$ dB, which results in a curve with a slope of $+40$ dB/decade. The phase angle approaches 180°.
- The magnitude asymptotes meet at the $0-$dB line when $\frac{\omega}{\omega_n}=1$.

The bode plots of a quadratic factor due to a pair of complex conjugate zeros are shown