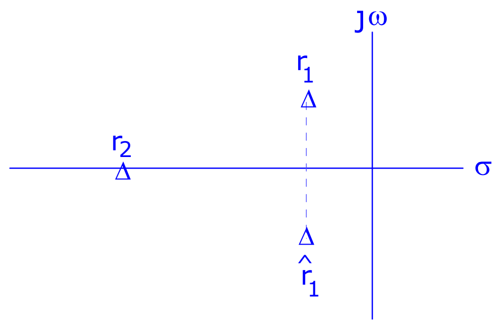
6. THE STABILITY OF LINEAR FEEDBACK SYSTEMS (CONT.)

THE RELATIVE STABILITY OF FEEDBACK CONTROL SYSTEM

The Routh-Hurwitz criterion ascertains the absolute stability of a system by determining whether any roots of the characteristic equation Lies in the right half of the s-plane. However, if the system satisfies the Routh-Hurwitz criterion and is absolutely stable, it is desirable to determine the relative stability; The relative stability is measured by the relative part of each root. Thus root r_2 is relatively more stable than r_1 , $\hat{r}_1$ as shown.



The Routh-Hurwitz criterion can be extended to ascertain relative stability. This can be accomplished by utilizing a change of variable, which shifts the s-plane axis in order to utilize the Routh-Hurwitz criterion.

Example

Determine the relative stability of the following characteristic equation
$s^3 + 4s^2 + 6s + 4 = 0$

Solution

As a first step, let $s_n = s + 2$. Applying the Routh-Hurwitz criterion will indicate if any of the roots of the characteristic equation is to the right of the line s=-2.

$(s_n - 2)^3 + 4(s_n - 2)^2 + 6(s_n - 2) + 4 = 0 \;\Rightarrow\; s_n^3 - 2s_n^2 + 2s_n = 0$
The necessary conditions are not satisfied.

Let us try $s_n = s + 1$. We obtain
$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = 0 \;\Rightarrow\; s_n^3 + s_n^2 + s_n + 1 = 0$

The Routh Array for this C.E. is:

s^3	1	1		
s^2	1	1		$U(s) = s^2 + 1$
s	~~0~~ ⇨ 2	~~0~~ ⇨ 0	←	$\frac{dU(s)}{ds} = 2s + 0$
s^0	1	0		0

There are no sign change in the first column. Hence all the roots have negative real parts less than -1except for a imaginary pair on the line $s=-1$, which are the roots of the auxiliary equation:

$s_n^2+1=0 \;\Rightarrow\; s_n=\pm j \;\Rightarrow s=-1\pm j$

Using MATLAB, one can find the actual roots of the C.E. :

-1, $-1\pm$j

The stability of State variable Systems

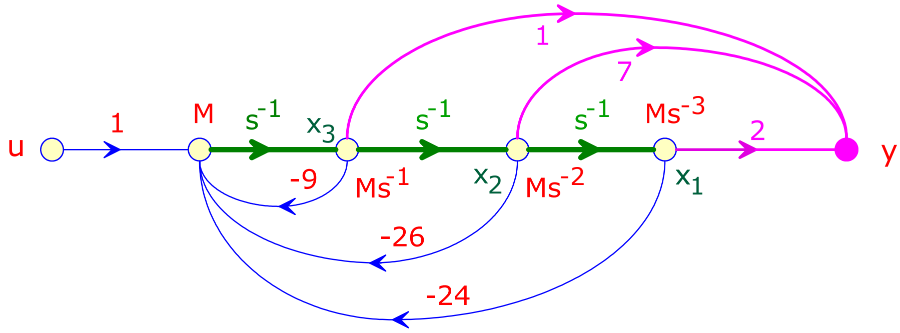
Consider the transfer function:

$$T(s)=\frac{Y(s)}{U(s)}=\frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

The CCF state variable representation is

$$\dot{\boldsymbol{x}}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}\boldsymbol{x}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \quad ;\; y=\begin{bmatrix} 2 & 7 & 1 \end{bmatrix}\boldsymbol{x}$$

The relevant SFG is



- The characteristic equation can be obtained from the denominator of the transfer function $T(s)$

  $q(s)=s^3+9s^2+26s+24=0$ or

- By evaluating the determinant $\Delta(s)$ of the SFG

  $\Delta(s)=1+9s^{-1}+26s^{-2}+24s^{-3}=0 \;\rightarrow\; s^3+9s^2+26s+24=0$ or

- Directly from the state equations by evaluating the determinant

$$|\lambda I-A|=0 \;\Rightarrow\; \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 24 & 26 & \lambda+9 \end{bmatrix}=0 \;\Rightarrow \lambda[\lambda(\lambda+9)+26]+24=0$$

$$=\lambda^3+9\lambda^2+26\lambda+24=0$$

Note: Any other state variable representations (e.g. OCF) will lead to the same characteristic equation. Try it!

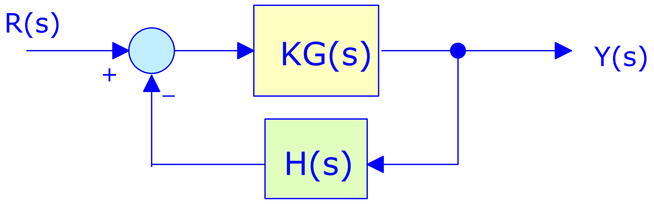
7. THE ROOT LOCUS METHOD

Introduction

The location of the closed-loop roots of the characteristic equation in the s-plane determines the relative stability and the transient response of the closed-loop control system. Therefore it is important to determine how the roots of the characteristic equation of a given system travel in the s-plane as the parameters are varied.

The root locus method, introduced by Evans in 1948, is a graphical method for sketching the locus of the roots in the s-plane as a parameter is varied.

The Root Locus Concept



Consider the simple single-loop control system shown .

The characteristic equation is

$$1 + KG(s)H(s) = 0$$

where K is a variable parameter.

The characteristic roots of the system must satisfy the characteristic equation. Because s is a complex variable, the characteristic equation may be rewritten in polar form as

$$|KG(s)H(s))| \angle G(s)H(s) = -1 + j0$$

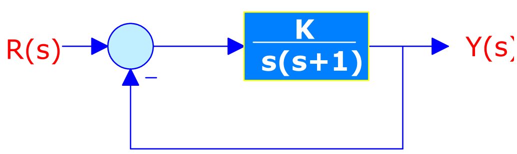
and therefore it is necessary that

$$|KG(s)H(s)| = 1$$ [Magnitude Condition]

$$\angle G(s)H(s) = -180^\circ$$ [Angle Condition]

Before presenting the root locus method and in order to give a clear idea of what a root locus plot looks like, we shall consider the system shown.

We shall obtain the characteristic equation analytically in terms of K and then vary K from 0 to ∞. It should be noted that this is not the proper way to construct the root locus plot. The proper way is by applying the general rules to be presented later.

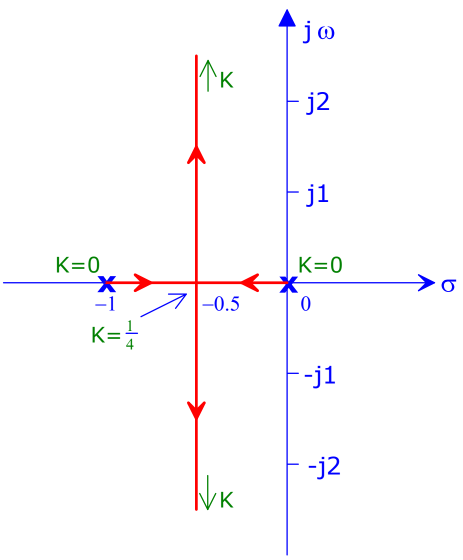


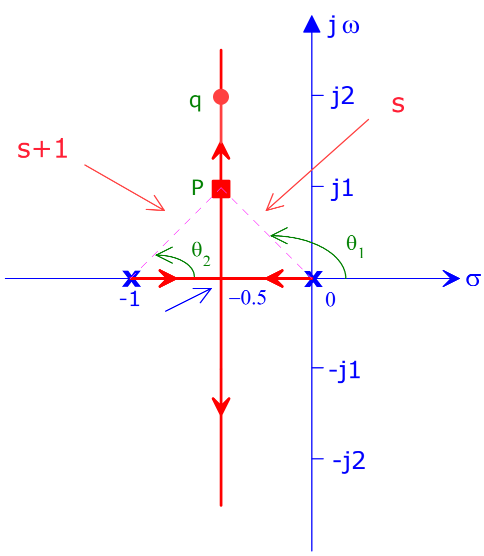
[If an analytical solution for the characteristic roots can be found easily, there is no need for the root locus method].

The characteristic equation representing this system is
$s^2 + s + K = 0$
The roots are
$s = -\frac{1}{2} \pm j\frac{1}{2}\sqrt{1-4K}$

1. The roots of the C.E. (i.e. closed-loop poles) corresponding to $K = 0$ are the same as the poles of the open-loop transfer function $G(s)H(s)$.
2. As the value of K is increased from 0 to $\frac{1}{4}$, the roots of the C.E. move toward point $(-\frac{1}{2}, 0)$. For values of $0 < K < \frac{1}{4}$, the roots of the C.E. are on the real axis. This corresponds to an overdamped system.
3. At $K = \frac{1}{4}$, the two characteristic roots unite. This corresponds to the case of a critically damped system.
4. As the value of K is increased from $\frac{1}{4}$, the characteristic roots of the C.E. break away from the real axis, becoming complex, and since the real part of the closed-loop poles (characteristic roots) is constant for $K > \frac{1}{4}$, the closed-loop poles move along the line $s = -\frac{1}{2}$. Hence, for $K > \frac{1}{4}$, the system becomes underdamped.

   The loci of the characteristic roots are plotted in the figure for all values of K





How to show that any point is on the root locus?

Use the angle condition

Consider point P on the root locus.

$\angle G(s)H(s) = \angle \frac{K}{s(s+1)} = -[\angle s + \angle s+1)] = -[\theta_1 + \theta_2] = \text{always } -180^\circ$

Hence, the point P is on the root locus.

How to find the gain at a specified point?

Use the magnitude condition

Consider point q on the root locus. It is required to find the gain at that point.

The coordinates of point q are : $-0.5+j2$

$$\left|\frac{K}{s(s+1)}\right|_{s=-0.5+j2} = \left|\frac{K}{(-0.5+j2)(-0.5+j2+1)}\right| = 1 \quad ; \rightarrow \; K = \frac{17}{4}$$

From the root locus plot, we clearly see the effects of changes in the value of K on the transient-response behavior of the second-order system.

- An increase in the value of K will decrease the damping ratio ζ, resulting in an increase of the overshoot of the response.
- An increase in the value of K will also result in increases in the damped and undamped frequencies.
- The system is always stable no matter how much K is increased.

Skill-Assessment Problem

Find the gain and the characteristic roots if the closed-loop response is to have a damping ratio of $\zeta = 0.707$

Answer: $K = 0.5, \; -0.5 \pm j0.5$