

6. THE STABILITY OF LINEAR FEEDBACK SYSTEMS (CONT.)

CASE 2. THERE IS A ZERO IN THE FIRST COLUMN, BUT SOME OTHER ELEMENTS OF THE ROW CONTAINING THE ZERO IN THE FIRST COLUMN ARE NONZERO;

If only the first element in one of the rows is zero, then we can replace the zero with a small positive constant $\epsilon > 0$ and proceed as before. We then apply the stability criterion by taking the limit as $\epsilon > 0$.

Example

Determine whether any of the roots of the characteristic equation $s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9 = 0$ are in the RHP.

Solution

The Routh Array for this C.E. is:

s^5	1	2	6
s^4	3	6	9
s^3	$0 \rightarrow \epsilon$	3	0
s^2	$(6\epsilon-9)/\epsilon$	9	0
s	$3 - 9\epsilon^2/(6\epsilon-9)$	0	
s^0	9		

s^5	+
s^4	+
s^3	+
s^2	-
s	+
s^0	+

The C.E. has **two** roots in the right-hand portion of the s-plane because there are **two** sign changes.

Using MATLAB, one can find the actual roots of the C.E. :

-2.9043
 $0.6567 \pm j1.2881$
 $-0.7046 \pm j0.9929$

This confirm the results of the Routh array

CASE 3. THERE IS A ZERO IN THE FIRST COLUMN, AND THE OTHER ELEMENTS OF THE ROW CONTAINING THE ZERO ARE ALSO ZERO

Another special case occurs when an entire row of the Routh array is zero. This condition occurs when the C.E. has roots symmetrically located about the origin of the s-plane {i.e. when factors such as $(s+\sigma)(s-\sigma)$ or $(s+j\omega)(s-j\omega)$ occur}.

- If the s^i row is zero, we form the following **auxiliary polynomial** $U(s)$ from the previous (nonzero) row in the Routh array:

$$U(s) = \beta_1 s^{i+1} + \beta_2 s^{i-1} + \beta_3 s^{i-3} + \dots$$
 Where $\{\beta_i\}$ are the coefficients of the s^{i+1} row in the Routh array.
- We then replace the s^i row by the coefficients of the **derivative** of the auxiliary polynomial, and complete the array.
- The auxiliary polynomial $U(s)$ is a factor of the characteristic equation $q(s)$.
- The order of the auxiliary polynomial $U(s)$ is always even and indicates the number of symmetrical roots.

Example

Determine whether any of the roots of the characteristic equation $s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 = 0$ are in the RHP.

Solution

The Routh Array for this C.E. is:

s^5	1	11	28	
s^4	5	23	12	
s^3	6.4	25.6		
s^2	3	12	→	$U(s) = 3s^2 + 12$
s	0 ⇒ 6	0 ⇒ 0	←	$\frac{dU(s)}{ds} = 6s + 0$
s^0	12			

There are no sign change in the first column. Hence all the roots have negative real parts except for a pair on the imaginary axis, which are the roots of the auxiliary equation:

$$3s^2 + 12 = 0 \Rightarrow s^2 + 4 = 0 \Rightarrow s = \pm j2$$

Using MATLAB, one can find the actual roots of the C.E. :

$$-3, -1, -1, \pm j2$$

This shows that the auxiliary equation is indeed a factor of the characteristic equation.

CASE 4. REPEATED ROOTS OF THE CHARACTERISTIC EQUATION ON THE $j\omega$ -AXIS

If the $j\omega$ -axis roots of the characteristic equation are simple, the system is neither stable nor unstable; it is instead called marginally stable, since it has an undamped sinusoidal mode. If the $j\omega$ -axis roots are repeated, the system response will be unstable, with a form $t\sin(\omega t + \phi)$. The Routh-Hurwitz criteria **will not** reveal this form of instability.

Example

Determine the stability of the system whose characteristic equation is $s^5 + s^4 + 2s^3 + 2s^2 + s + 1 = 0$

Solution

The Routh Array for this C.E. is:

s^5	1	2	1	
s^4	1	2	1	$U(s) = s^4 + 2s^2 + 1$
s^3	$\theta \Rightarrow 4$	$\theta \Rightarrow 4$		$\frac{dU(s)}{ds} = 4s^3 + 4s$
s^2	1	1		$U(s) = s^2 + 1$
s	$\theta \Rightarrow 2$	$\theta \Rightarrow 0$		$\frac{dU(s)}{ds} = 2s$
s^0	1			

Note the absence of sign changes, a condition that falsely indicates that the system is marginally stable. The impulse response of the system increases with time as $t\sin(\omega t + \phi)$.

The auxiliary polynomials at the s^2 line is $s^2 + 1$

The auxiliary polynomials at the s^4 line is $s^4 + 2s^2 + 1 = (s^2 + 1)^2$; indicating the repeated roots on the $j\omega$ -axis.

DRILL PROBLEM [DUE : Wednesday 5 November 2003]

For the control system shown:

- Determine the value of K when the system oscillates, and the frequency of oscillations.
- Calculate the roots of the closed-loop system for the K determined in (a).
- Simulate the system for the K determined in (a). Compare the frequency of oscillations with (a).

