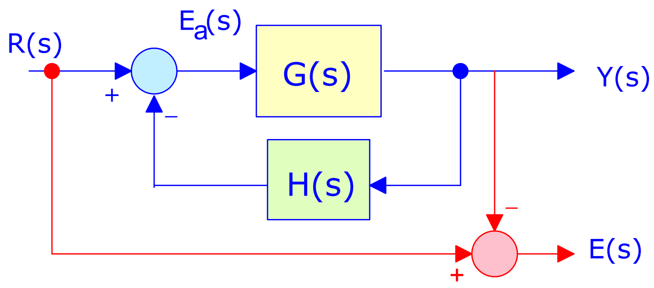
5. THE PERFORMANCE OF FEEDBACK CONTROL SYSTEMS

STEADY-STATE ERROR OF FEEDBACK CONTROL SYSTEMS

Let us consider the closed-loop feedback control system shown below.

- The system actuating signal, which is a measure of the system error, is denoted by $E_a(s)$.
- However, the actual system error is $E(s) = R(s) - Y(s)$



$$E(s) = R(s) - \frac{G(s)}{1+G(s)H(s)}R(s)$$

$$E(s) = \frac{[1+G(s)H(s) - G(s)]}{1+G(s)H(s)}R(s)$$

Note that the system error $E(s)$ is equal to $E_a(s)$ if $H(s) = 1$.

or,

$$E(s) = R(s) - T(s)R(s) = R(s)[1 - T(s)] \quad ; \quad T(s) = \frac{G(s)}{1+G(s)H(s)}$$

The steady-state error is then

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{[1+G(s)H(s)-G(s)]}{1+G(s)H(s)}R(s) = \lim_{s\to 0} sR(s)[1-T(s)]$$

STEADY-STATE ERROR OF UNITY-FEEDBACK CONTROL SYSTEMS

The system error, when $H(s) = 1$, is

$$E(s) = \frac{1}{1+G(s)}R(s)$$

The steady-state error when $H(s) = 1$,is then

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

It is useful to determine the steady-state error of the system for the three standard test inputs for a unity-feedback system. Let us assume that the open-loop transfer function is written in general form as

$$G(s) = K\frac{\prod_{i=1}^{M}(s+z_i)}{s^N \prod_{k=1}^{Q}(s+p_k)}$$

The number of integrators is equal to N.
For $N = 0$, we have a type-zero system
For $N = 1$, we have a type-one system
For $N = 2$, we have a type-two system

Let us also use the following notations:

$\lim_{s\to 0} G(s) = K_P$ position error constant

$\lim_{s\to 0} sG(s) = K_v$ velocity error constant

$\lim_{s\to 0} s^2 G(s) = K_a$ acceleration error constant

Step Input. $R(s) = \frac{A}{s}$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{A}{1+G(s)} = \frac{A}{1+\lim_{s\to 0} G(s)} = \frac{A}{1+K_p}$$

N	K_p	$\mathbf{e_{ss}}$
0	$K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k$	$\frac{A}{1+K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k}$
≥ 1	∞	0

Ramp Input. $R(s) = \frac{A}{s^2}$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{A}{s(1+G(s))} = \lim_{s\to 0} \frac{A}{s+sG(s)} = \frac{A}{\lim_{s\to 0} sG(s)} = \frac{A}{K_v}$$

N	K_v	$\mathbf{e_{ss}}$
0	0	∞
1	$K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k$	$\frac{A}{K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k}$
≥ 2	∞	0

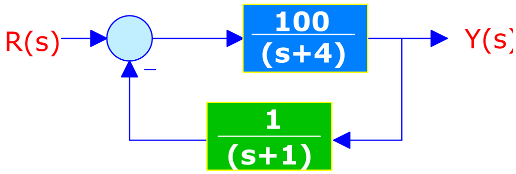
Acceleration Input. $R(s) = \frac{A}{s^3}$

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{A}{s^2(1+G(s))} = \lim_{s\to 0} \frac{A}{s^2+s^2G(s)} = \frac{A}{\lim_{s\to 0} s^2G(s)} = \frac{A}{K_a}$$

N	K_a	$\mathbf{e_{ss}}$
0	0	∞
1	0	∞
2	$K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k$	$\frac{A}{K\prod_{i=1}^{M} z_i / \prod_{k=1}^{Q} p_k}$
≥ 3	∞	0

Example

Find the steady-state error for a unit-step input.



Solution # 1

$$E(s) = R(s) - Y(s)$$

$$E(s) = R(s) - \frac{G(s)}{1 + G(s)H(s)} R(s)$$

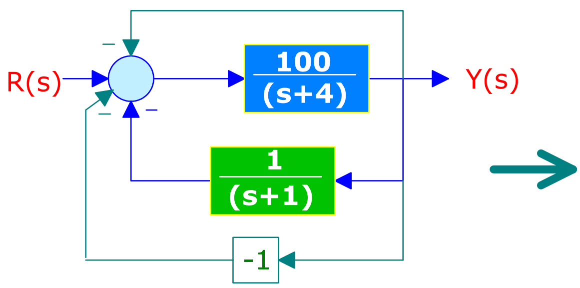
$$E(s) = \frac{[1 + G(s)H(s) - G(s)]}{1 + G(s)H(s)} R(s)$$

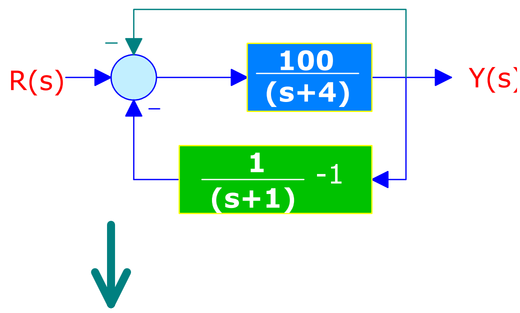
$$E(s) = \frac{[1 + \frac{100}{s+4}\frac{1}{s+1} - \frac{100}{s+4}]}{1 + \frac{100}{s+4}\frac{1}{s+1}} \frac{1}{s} = \frac{(s+4)(s+1) + 100 - 100(s+1)}{(s+4)(s+1) + 100} \frac{1}{s}$$

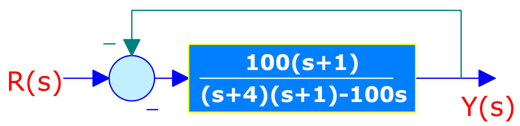
$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{4}{104} = 3.85\%$$

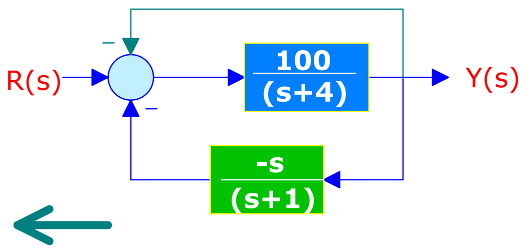
Solution # 2

The basic idea is to convert the system to a unity-feedback system by adding and subtracting unity feedback paths, and use the relevant formulas.









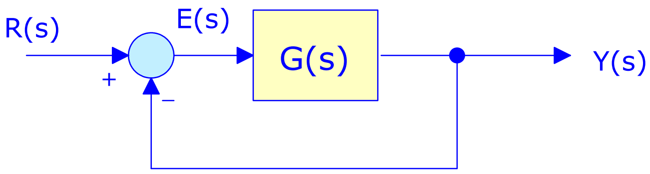
$$K_P = \lim_{s \to 0} G(s) = \frac{100}{4}$$

$$e_{ss} = \frac{1}{1 + K_p} = \frac{1}{1 + \frac{100}{4}} = \frac{4}{104} = 3.85\%$$

Same Answer

Example

A unity feedback system has the following forward transfer function



$$G(s) = 1000\frac{(s+8)}{(s+7)(s+9)}$$

Find the steady-state error for
the standard unit step, unit ramp, and unit parabolic inputs.

Solution

First we see if the system is stable or not.

C.E. ➔ $s^2 + 1016s + 8063 = 0$; [stable]

Type 0 system

$$\lim_{s \to 0} G(s) = K_P = \tfrac{8000}{63} \;;\; \lim_{s \to 0} sG(s) = K_v = 0 \;;\; \lim_{s \to 0} s^2 G(s) = K_a = 0 \;;$$

$$e_{ss}|_{step} = \frac{1}{1+K_p} = \frac{1}{1+\frac{8000}{63}} = \frac{63}{8001} = 0.787\%$$

$$e_{ss}|_{ramp} = \frac{1}{K_v} = \infty$$

$$e_{ss}|_{parabolic} = \frac{1}{K_a} = \infty$$