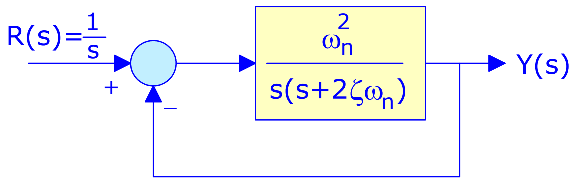
5. THE PERFORMANCE OF FEEDBACK CONTROL SYSTEMS

THE S-PLANE ROOT LOCATION AND THE TRANSIENT RESPONSE

Let us consider a generalized single loop second order system.
The closed-loop output is

$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)$$



With a unit step input, we obtain

$$Y(s)=\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}=\frac{\omega_n^2}{s\left(s+\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}\right)}=\frac{\omega_n^2}{s\left(s+\alpha\pm j\omega\right)}$$

$\alpha=\zeta\omega_n$ is the damping factor, and $\omega=\omega_n\sqrt{1-\zeta^2}$ is the damped frequency

$$Y(s)=\frac{1}{s}+\frac{K_1}{s+\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}}+\frac{K_2}{s+\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}}$$

$$K_1=\frac{-\sqrt{1-\zeta^2}+j\zeta}{2\sqrt{1-\zeta^2}}\;;\;K_2=\frac{-\sqrt{1-\zeta^2}-j\zeta}{2\sqrt{1-\zeta^2}}$$

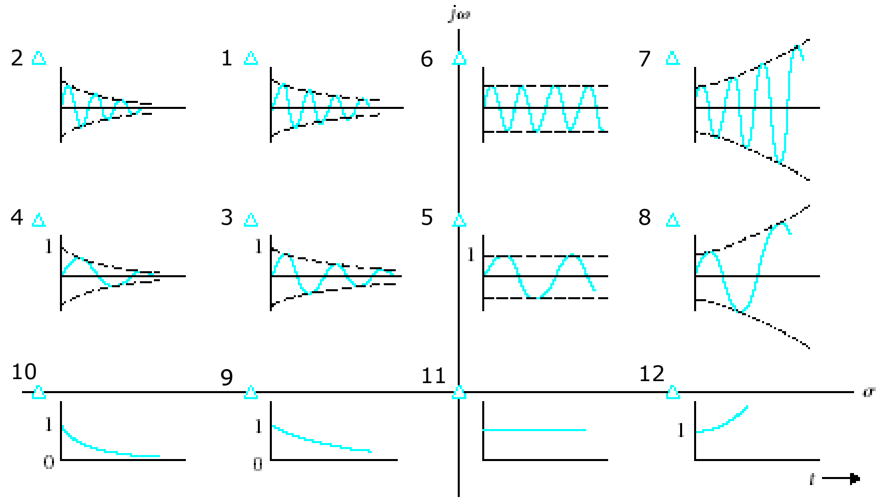
$$y(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t+\theta)\;;\theta=\cos^{-1}\zeta$$

The roots of the system are complex conjugates. The transient response is composed of the steady-state output, and a damped sinusoidal term.

For the response to be stable, one must require that the real part of the roots , $-\zeta\omega_n$, be in the left-hand portion of the s-plane.

It is very important to understand the relationship between the complex-frequency representation of a linear system, through the poles and zeros of its transfer function, and its time-domain response to step and other inputs.

The impulse response for various root locations in the s-plane is shown in the figure. The location discloses important information about the particular response.



Observations

1. From s-plane, 1, 2, 6, and 7 have the same damped frequency, say ω_1 You can see that all relevant responses have the same frequency which is equal to ω_1 .
2. From s-plane, 4, 3, 5, and 8 have the same damped frequency, say ω_2You can see that all relevant responses have the same frequency which is equal to ω_2 .
3. From s-plane, it is clear that $\omega_2 < \omega_1$. This can be seen from the impulse responses as well.
4. From s-plane, (2 and 4) have the same damping factor. Hence they have the same settling time, say T_{s1}.
5. From s-plane, (1 and 3) have the same damping factor. Hence they have the same settling time, say T_{s2}.
6. From s-plane, it is clear that $T_{s2} > T_{s1}$. This can be verified from the impulse responses as well.
7. From s-plane, 5, and 6 have zero damping factor, which means infinite settling time. The response is an undamped sinusoidal.
8. From s-plane, 1,2 3, and 4 have negative real parts, which means system is stable. It is clear that all relevant responses are decaying with time.
9. From s-plane, 7,8, and 12 have positive real parts. We expect the impulse responses to increase with time, which they are.
10. First order systems, 9, and 10 are both stable, but time constant of 10 is smaller than that of 9. {time constant equal reciprocal of real part}. This is verified from impulse responses.
11. First order system 11, has zero real part, hence the time constant is infinity. This is also verified from the responses.

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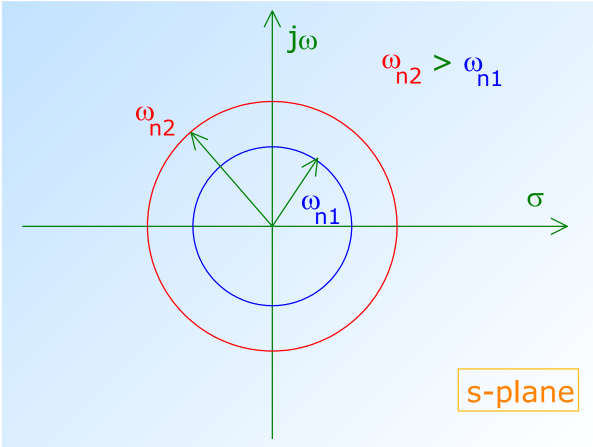
CONSTANT- ω_n LOCI

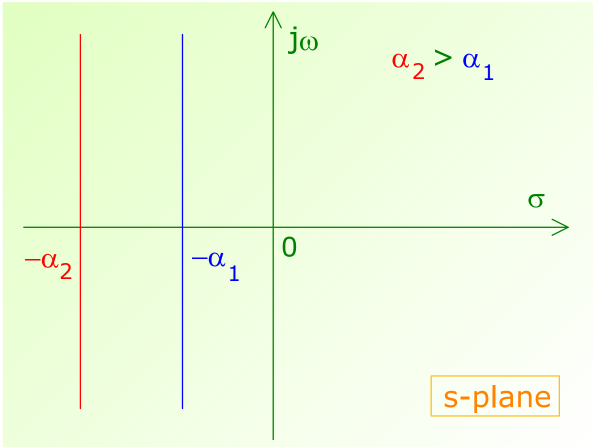
The roots of the characteristic equation of second-order systems which have the same natural frequency lie on a circle whose center is at the origin and with radius equal to ω_n.



> Constant-ω_n locus in the s-plane is a circle whose center is the origin and radius $= \omega_n$.

CONSTANT- α LOCI

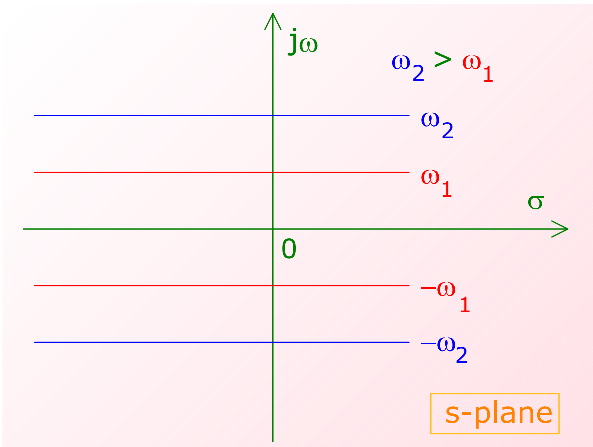
The roots of the characteristic equation of second-order systems which have the same damping factor α lie on a vertical straight line passing through the point $(-\alpha, 0)$.



> Constant-α locus in the s-plane is a vertical straight line passing through the point $(-\alpha, 0)$.

CONSTANT- ω LOCI

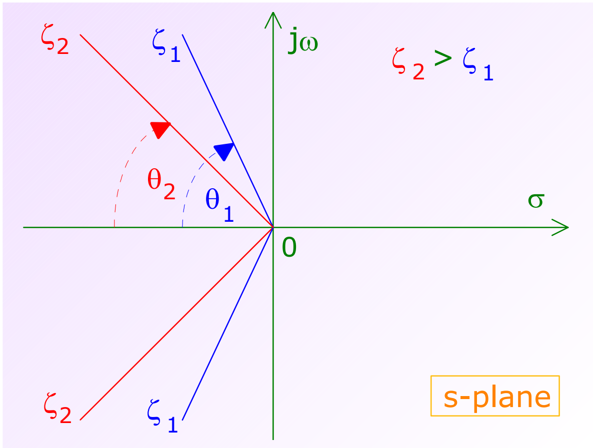
The roots of the characteristic equation of second-order systems which have the same damped frequency ω lie on a horizontal straight line that passes through the point $(0, \omega)$, and its mirror image..



> Constant-ω loci in the s-plane is an horizontal straight line that passes through the point $(0, \omega)$, and its mirror image.

CONSTANT- ζ LOCI

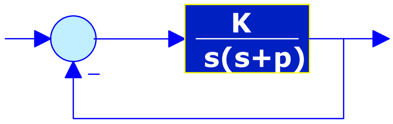
The roots of the characteristic equation of second-order systems which have the same damping ratio ζ lie on straight lines whose slope are $\theta = \cos^{-1}\zeta$,and its mirror image.



Constant-ζ locus in the s-plane is a straight line whose slope is $\theta = \cos^{-1}\zeta$, and its mirror image.

Example **DESIGN**

We desire to select the gain K and the parameter p so that the following specifications are satisfied:



1. $P.O. < 4.3\%$
2. $T_s \leq 4s$

SOLUTION

- The damping ratio ζ for an overshoot of 4.3% is:

$$\zeta = \frac{-\ln(\frac{P.O.}{100})}{\sqrt{\pi^2 + \ln^2(\frac{P.O.}{100})}} = \frac{-\ln(\frac{4.3.}{100})}{\sqrt{\pi^2 + \ln^2(\frac{4.3.}{100})}} = 0.707$$

This damping ratio is shown graphically as a line in the figure.

- A settling time $T_s \leq 4$ implies that:
$$\frac{4}{\zeta\omega_n} < 4 \;\Rightarrow\; \zeta\omega_n \geq 1$$

This region is also shown in the figure. The region that satisfies both requirements is shown in yellow.

You can select the closed-loop poles anywhere in this region. Say at r_1. In this case:

$\omega_n = \sqrt{2}$; $\zeta = \frac{1}{\sqrt{2}}$

$K = \omega_n^2 = 2$; $P = 2\zeta\omega_n = 2$