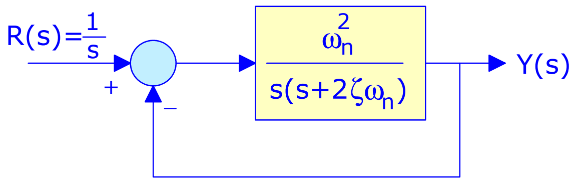
5. THE PERFORMANCE OF FEEDBACK CONTROL SYSTEMS

Performance of a 2nd Order Systems

Let us consider a generalized single loop second order system and determine its response to a unit-step input.



The closed-loop output is

$$Y(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s)$$

With a unit step input, we obtain

$$Y(s)=\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}=\frac{\omega_n^2}{s\left(s+\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}\right)}$$

Where

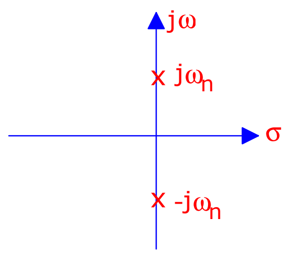
ω_n Natural frequency (the frequency of oscillation of the system without damping.

ζ Damping ratio

A second-order system exhibits a wide range of responses based on the values of the parameters ζ and ω_n.We will now explain each response and show how we can directly use the poles to determine the nature of the response.

Case 1 Undamped Response ($\zeta=0$)

$$Y(s)=\frac{\omega_n^2}{s\left(s^2+\omega_n^2\right)}=\frac{\omega_n^2}{s\left(s\pm j\omega_n\right)}$$



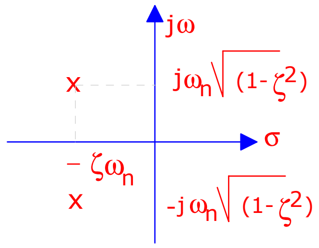
Poles: 2 imaginary at $\pm j\omega_n$

$y(t)=1-\cos(\omega_n t)$

Undamped sinusoid with radian frequency equal to the imaginary part of the poles.

Case 2 Underdamped Response $(0<\zeta<1)$

$$Y(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{\omega_n^2}{s\left(s+\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}\right)}$$



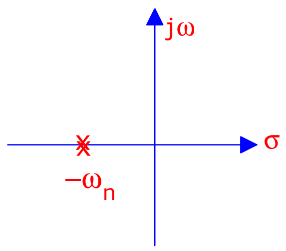
Poles: 2 complex at $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$y(t)=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t+\theta)\ ;\ \theta=\cos^{-1}\zeta$$

Damped Sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid ω, is equal to the imaginary part of the pole.

Case 3 Critically damped Response $(\zeta=1)$

$$Y(s)=\frac{\omega_n^2}{s(s^2+2\omega_n s+\omega_n^2)}=\frac{\omega_n^2}{s(s+\omega_n)^2}$$



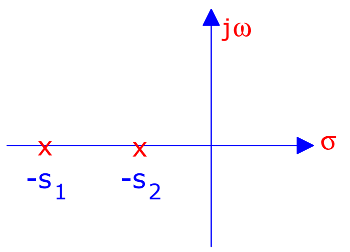
Poles: 2 real at $-\omega_n$

$$y(t)=1-e^{-\omega_n t}(1+\omega_n t)$$

One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location.

Case 4 Overdamped Response $(\zeta>1)$

$$Y(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{\omega_n^2}{s\left(s+\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}\right)}$$

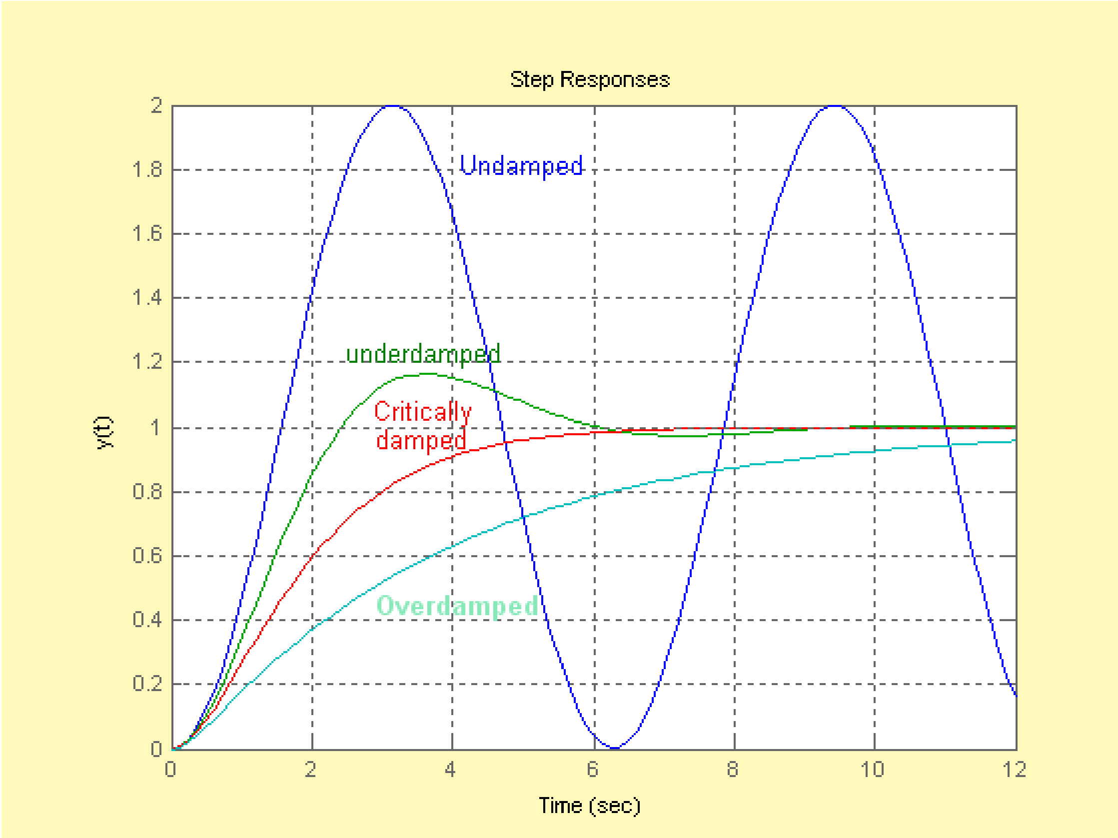


Poles: 2 real at $-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

$$y(t)=1+\frac{\omega_n}{2\sqrt{1-\zeta^2}}\left(\frac{e^{-s_1 t}}{s_1}-\frac{e^{-s_2 t}}{s_2}\right);$$
$$s_1=\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\ ,\ s_2=\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}$$

Two exponential terms with time constants equal to the reciprocal of the pole locations

The step responses for the four cases of damping are superimposed in the figure shown. Notice that the critically damped case is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.



Transient-Response Specifications

The transient response of a practical control system often exhibits damped oscillations before reaching steady-state (i.e. underdamped response). In specifying the transient response characteristics of a control system to a unit-step input, it is common to specify the following standard performance measures:

Rise Time T_r

Ti me required for the response to rise from 10 to 90% or 0 to 100% of its final value. For underdamped second-order systems, the 0 to 100% rise time is normally used. For overdamped systems, the 10 to 90% rise time is commonly used.

Peak Time T_p

Time required for the response to reach the first peak of the overshoot.

Percent Overshoot *P.O.*

The peak value of the time response M_{p_t} measured from the final value f_v of the response . Normally f_v is the magnitude of the input, but many systems have a final value different from the desired input magnitude.

$$P.O. = \frac{M_{p_t} - f_v}{f_v} 100\,\%$$

Settling Time T_s

Time required for the response to settle within a certain percentage δ of the input amplitude. (Usually 2% or 5%). We seek to determine the time, T_s, for which the response remains within 2% of the final value. This occurs approximately when

$$e^{-\zeta\omega_n T_s} < 0.02 \;\Rightarrow\; \zeta\omega_n T_s = 4 \;\rightarrow\; T_s = \frac{4}{\zeta\omega_n}$$

Hence we will define the settling time as four time constants (that is $\tau = \dfrac{1}{\zeta\omega_n}$) of the dominant roots of the Characteristic equation.

The above performance measures are shown in the figure.