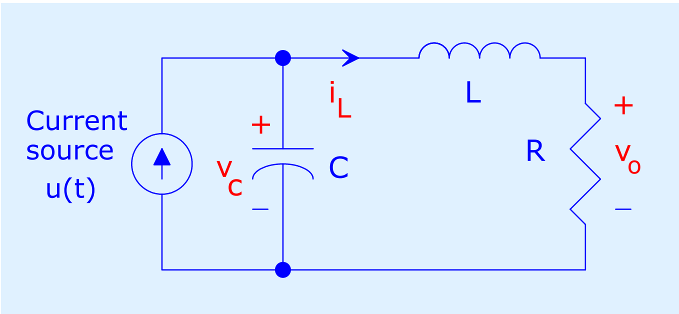
# 3. **STATE VARIABLE MODELS (cont.)**

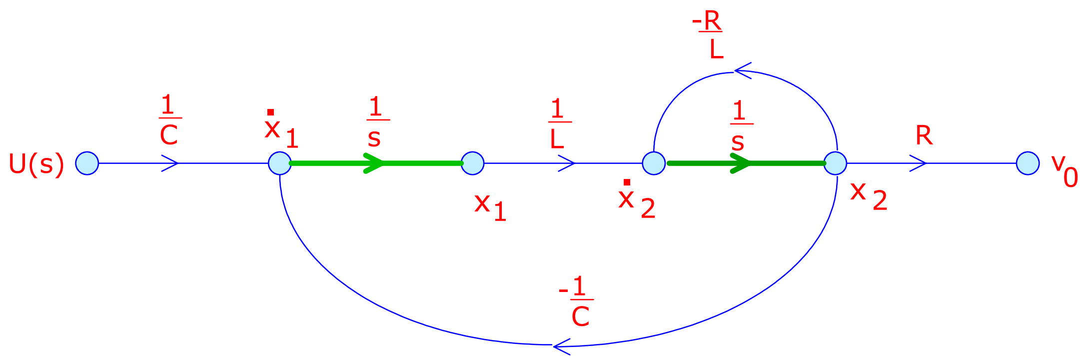
## SIGNAL-FLOW GRAPH STATE MODELS

It is useful to develop a state flow graph model of the state differential equation and use this model to relate the state variable concept to the familiar transfer function representation. As an example, consider the RLC circuit for which we wrote the state equations earlier




$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$
$$\frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$
$$y = v_0 = Rx_2$$

The flow graph representing these simultaneous equations is shown. First, assign a node to each state variable, each derivative of state variable, input, and output . The node of a state variable derivative is then connected to the node of the corresponding state variable by a branch with a gain of $s^{-1}$(i.e. Integrator). Finally, the inputs to the nodes of the state variables derivative , and to the output node are determined from the state equations.




Using Mason's signal-flow gain formula, we may obtain the transfer function

$$\frac{V_0(s)}{U(s)} = \frac{\frac{1}{C}\frac{1}{s}\frac{1}{L}\frac{1}{s}R}{1+\frac{R}{L}\frac{1}{s}+\frac{1}{C}\frac{1}{s}\frac{1}{L}\frac{1}{s}} = \frac{\frac{R}{LC}}{s^2+\left(\frac{R}{L}\right)s+\left(\frac{1}{LC}\right)}$$

## Converting a Transfer Function to State Space

### Part I    Constant term in numerator [no numerator dynamics]

Consider the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

The corresponding differential equation is found by taking the Laplace transform of the transfer function

$$\dddot{y} + 9\ddot{y} + 26\dot{y} + 24y = 24u$$

Let $\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \end{aligned}$ then $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x3 + 24u \end{aligned}$

In vector-matrix form

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} u \quad ; \; y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \boldsymbol{x}$$

Notice that the third row of the system matrix has the same coefficients as the denominator of the transfer function but negative and in reverse order

**Drill problem** **[due on Monday 21 October, 2002]**

1. Find a state-space representation of the system whose transfer function is

$$\frac{Y(s)}{U(s)} = \frac{100}{s^4 + 20s^3 + 10s^2 + 7s + 100}$$

2. Repeat part 1 Using MATLAB . [tf2ss]
3. Draw a block diagram and a SFG for the state equations obtained in part 1.

## PART II Polynomial in numerator

### Method 1 Phase variable format ( Control canonical form CCF)

Consider the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

- Divide numerator and denominator by $s^3$ [highest power of $s$in denominator.

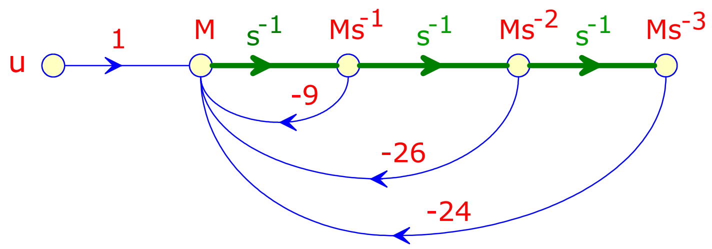
$$\frac{Y(s)}{U(s)} = \frac{s^{-1}+7s^{-2}+2s^{-3}}{1+9s^{-1}+26s^{-2}+24s^{-3}}$$

- Multiply the numerator and the denominator by $M(s)$

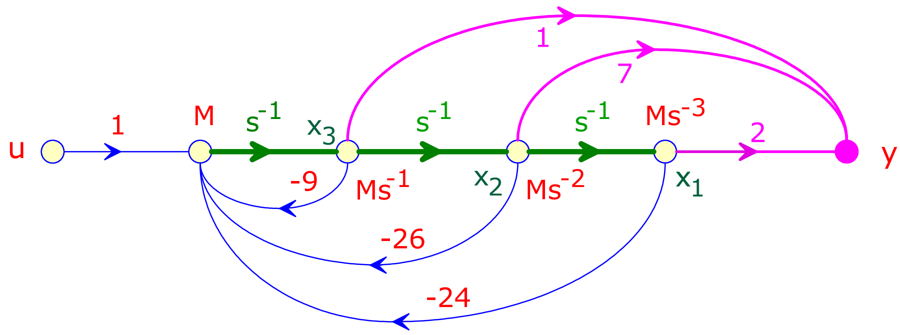
$$\frac{Y(s)}{U(s)} = \frac{s^{-1}+7s^{-2}+2s^{-3}}{1+9s^{-1}+26s^{-2}+24s^{-3}}\frac{M(s)}{M(s)}$$

- Draw a SFG for the denominator

$$U(s) = [1+9s^{-1}+26s^{-2}+24s^{-3}]\,M(s)$$
$$M = U-9[s^{-1}M]-26[s^{-2}M]-24[s^{-3}M]$$




- Draw a SFG for the numerator
$Y(s) = [s^{-1}M]+7[s^{-2}M]+2[s^{-3}M]$




- Select the state variables as shown

Hence, the state equation is:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad ; \; y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \boldsymbol{x}$$

Notice that the third row of the system matrix has the same coefficients as the denominator of the transfer function but negative and in reverse order. Also, the output matrix C consists of the coefficients of the numerator in reverse order.

Note: **This holds if the order of the numerator is less than that of the denominator.**

Skill-Assessment

Find the state equations and output equation in CCF, by observation, for the transfer function:

$$\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+7s+9}$$

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