4.29

a) $\vec{E} = -\nabla V = -(2x\,\vec{a}_x + 4y\,\vec{a}_y + 8z\,\vec{a}_z)$

c) $\vec{E} = -\nabla V = -\left[2\rho(z+1)\sin\phi\,\vec{a}_\rho + \rho(z+1)\cos\phi\,\vec{a}_\phi + \rho^2\sin\phi\,\vec{a}_z\right]$

4.30

a) $V_P = V_{1P} + V_{2P} + V_{3P}$

$$V_i = \frac{Q_i}{4\pi\epsilon_0 R_i}$$

$$V_{1P} = \frac{10^{-3}}{\left(\frac{10^{-9}}{9}\right)(\sqrt{6})} = 3.67\,MV$$

$$V_{2P} = \frac{-2\times10^{-3}}{\left(\frac{10^{-9}}{9}\right)\sqrt{18}} = -4.24\,MV$$

$$V_{3P} = \frac{3\times10^{-3}}{\left(\frac{10^{-9}}{9}\right)\sqrt{57}} = 3.58\,MV$$

$\therefore V_P = (3.67 - 4.24 + 3.58)\,MV = 3.01\,MV$

b) $V_{PQ} = V_Q - V_P$

$$V_Q = \frac{10^{-3}}{\left(\frac{10^{-9}}{9}\right)\sqrt{6}} - \frac{2\times10^{-3}}{\left(\frac{10^{-9}}{9}\right)\sqrt{22}} + \frac{3\times10^{-3}}{\left(\frac{10^{-9}}{9}\right)\sqrt{49}}$$

$= 3.69\,MV$

$\therefore V_{PQ} = (3.69 - 3.01)\,MV = 0.68\,MV$

4.32

In principle we can use $V = \int_V \frac{\rho_v \, dv}{4\pi\epsilon_0 R}$ to find V. However, this method is difficult to apply in this case.

Alternatively, we can use Gauss's law to find $\vec{E}$, then we can find V from $\vec{E}$.

$r < a$

$$\oint \vec{D} \cdot \vec{ds} = Q = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} \left(\rho_0 \frac{r}{a}\right) r^2 \sin\theta \, d\theta \, d\phi \, dr = \int_V \rho_v \, dv$$

$$\therefore D \, 4\pi r^2 = \rho_0 \left(\frac{1}{a}\right) \frac{r^4}{4} (2\pi)(2)$$

$$\vec{D} = \frac{(\rho_0/a) r^2}{4} \vec{a}_r$$

$$\vec{E} = \frac{\rho_0 r^2}{4a\epsilon_0} \vec{a}_r , \quad (r < a)$$

$r > a$

$$\oint \vec{D} \cdot \vec{ds} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \left(\rho_0 \frac{r}{a}\right) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$D \, 4\pi r^2 = \rho_0 \left(\frac{1}{a}\right)\left(\frac{a^4}{4}\right)(2\pi)(2)$$

$$\vec{D} = \frac{\rho_0 a^3}{4 r^2} \vec{a}_r$$

$$\vec{E} = \frac{\rho_0 a^3}{4\epsilon_0 r^2} \vec{a}_r , \quad (r > a)$$

$$\therefore \vec{E} = \begin{cases} \dfrac{\rho_o r^2}{4\epsilon_o a} \vec{a}_r & , r < a \\ \dfrac{\rho_o a^3}{4\epsilon_o r^2} \vec{a}_r & , r > a \end{cases}$$

$$V_p = -\int_{\infty}^{p} \vec{E} \cdot \vec{dl} \qquad \text{[reference point at } \infty\text{]}$$

For $r > a$

$$V_p = -\int_{\infty}^{r} \frac{\rho_o a^3}{4\epsilon_o r^2} dr = \left.\frac{\rho_o a^3 r^{-1}}{4\epsilon_o}\right|_{\infty}^{r} = \frac{\rho_o a^3}{4\epsilon_o r}$$

For $r < a$

$$V_p = -\int_{\infty}^{p} \vec{E} \cdot \vec{dl} = -\int_{\infty}^{a} \frac{\rho_o a^3}{4\epsilon_o r^2} dr - \int_{a}^{r} \frac{\rho_o r^2}{4\epsilon_o a} dr$$

$$= \left.\frac{\rho_o a^3 r^{-1}}{4\epsilon_o}\right|_{\infty}^{a} - \left.\frac{\rho_o r^3}{12\epsilon_o a}\right|_{a}^{r}$$

$$= \frac{\rho_o a^2}{4\epsilon_o} - \frac{\rho_o r^3}{12\epsilon_o a} + \frac{\rho_o a^2}{12\epsilon_o}$$

$$= \frac{\rho_o (4a^3 - r^3)}{12\epsilon_o a}$$

$$\therefore V = \begin{cases} \rho_o (4a^3 - r^3)/12\epsilon_o a & , r < a \\ \dfrac{\rho_o a^3}{4\epsilon_o r} & , r > a \end{cases}$$

4.34

a) Since the two charges are equal and are at equal distance from the center of the loop, then:

$$V_{center} = 2 \times \frac{Q/2}{4\pi\epsilon_0 (4)} = \frac{Q}{16\pi\epsilon_0} = \frac{60 \times 10^{-6}}{16\pi\left(\frac{10^{-9}}{36\pi}\right)} = 135 KV$$

b) $$V = 3 \times \frac{Q/3}{4\pi\epsilon_0 (4)} = \frac{Q}{16\pi\epsilon_0} = 135 KV$$

c) $V = 135 KV$, since Q is divided continuously and uniformly over the loop.

Another method:

$$V = \int_l \frac{\rho_l \, dl}{4\pi\epsilon_0 R} = \int_{\phi=0}^{2\pi} \frac{\frac{Q}{8\pi} \rho \, d\phi}{4\pi\epsilon_0 \rho} \Bigg|_{\rho=4} = \frac{Q}{16\pi\epsilon_0} = 135 KV$$

4.39

Refer to eqn. (4.78).

$$V = \frac{Q}{4\pi\epsilon_0} \frac{d\cos\theta}{r^2}$$ [valid for a dipole aligned along the z-axis, see figure 4.20].

This equation can be generalized, so that it can be used for an arbitrary orientation of the dipole.

$$V = \frac{(Qd)\cos\theta}{4\pi\epsilon_0 r^2} = \frac{Q(\vec{d}\cdot\vec{a_r})}{4\pi\epsilon_0 r^2}$$

where $\vec{d}$ is a vector with a magnitude of d and directed from $-Q$ to $+Q$.

Note that $\vec{d}\cdot\vec{a_r} = d\cos\theta$ in this case.

For the current problem $\vec{d} = d\,\vec{a_y}$

$$\therefore\ V = \frac{Q(d\,\vec{a_y}\cdot\vec{a_r})}{4\pi\epsilon_0 r^2} = \frac{Qd\,[\sin\theta\sin\phi]}{4\pi\epsilon_0 r^2}$$

$$= \frac{Qd\sin\theta\sin\phi}{4\pi\epsilon_0 r^2}$$

$$\vec{E} = -\nabla V = -\left[-\frac{Qd\sin\theta\sin\phi}{2\pi\epsilon_0 r^3}\vec{a_r} + \frac{Qd\cos\theta\sin\phi}{4\pi\epsilon_0 r^3}\vec{a_\theta} + \frac{Qd\cos\phi}{4\pi\epsilon_0 r^3}\vec{a_\phi}\right]$$

$$= \frac{Qd\sin\theta\sin\phi}{2\pi\epsilon_0 r^3}\vec{a_r} - \frac{Qd\cos\theta\sin\phi}{4\pi\epsilon_0 r^3}\vec{a_\theta} - \frac{Qd\cos\phi}{4\pi\epsilon_0 r^3}\vec{a_\phi}$$