Chapter 2: The Random Variable

The outcome of a random experiment need not be a number, for example tossing a coin or selecting a color ball from a box.

However we are usually interested not in the outcome itself, but rather in some measurement or numerical attribute of the outcome.

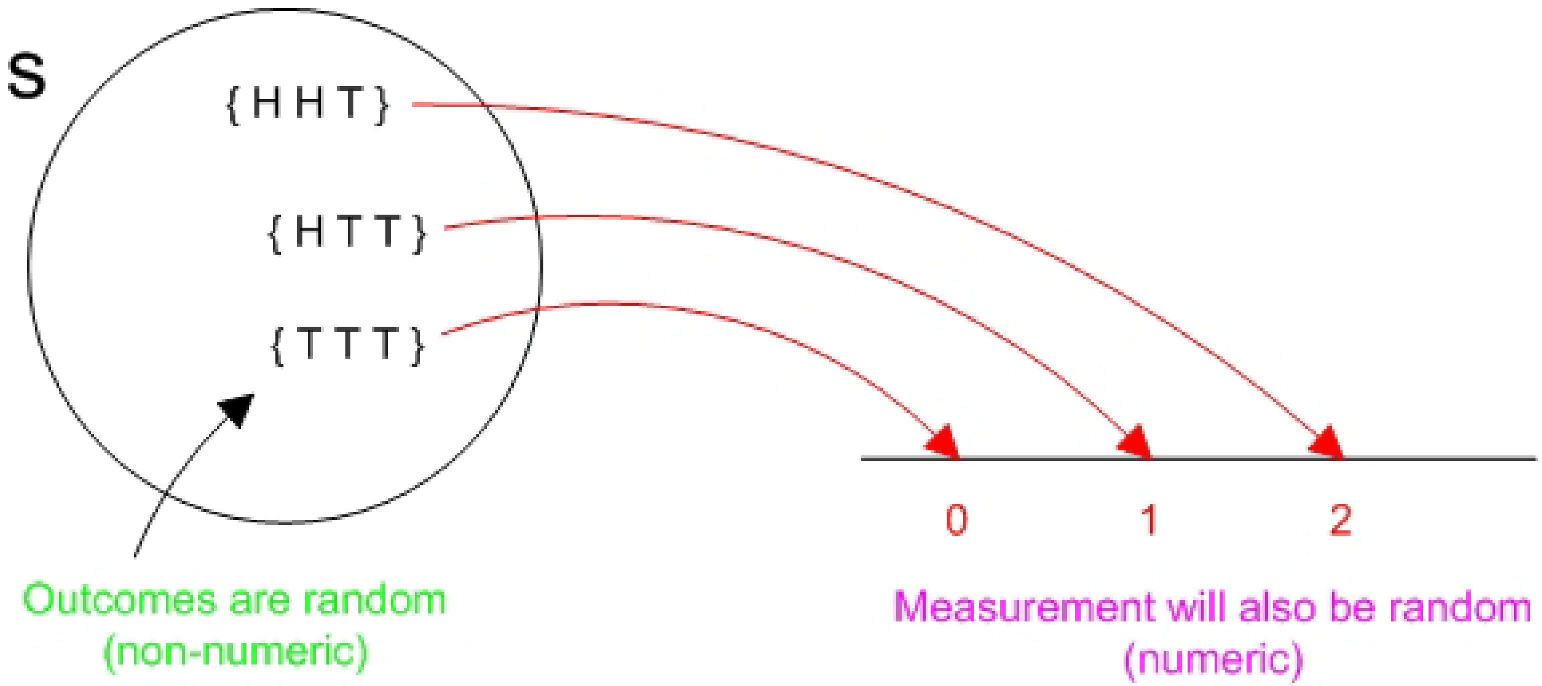
Examples

In tossing a coin we may be interested in the total number of heads and not in the specific order in which heads and tails occur.

In selecting a student name from an urn (box) we may be interested in the weight of the student.

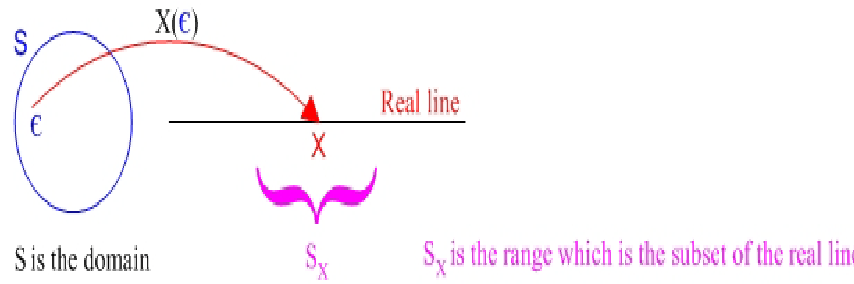
In each of these examples, a numerical value is assigned to the outcome.

Consider the experiment of tossing a coin 3 times and observing the number of “**Heads**” (which is a numeric value).



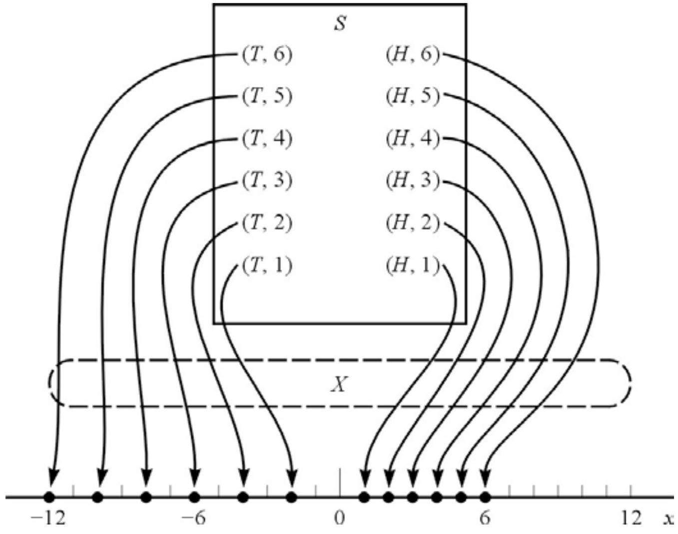
The Random Variable Concept

A random variable X is a function that assigns a real number X(ϵ) to each outcome ϵ in the sample space.



Example 2.1-1: The experiment of rolling a die and tossing a coin

Let X be a random variable that maps the "**head**" outcome to the positive number which correspond to the dots on the die, and "**tail**" outcome map to the negative number that are equal in magnitude to *twice* the number which appear on the die.



The Random Variable Concept

Random variables can be

Discrete

Flipping a coin 3 times and counting the number of heads.

Selecting a number from the positive integers.

Number of cars arriving at gas station A.

Continuous

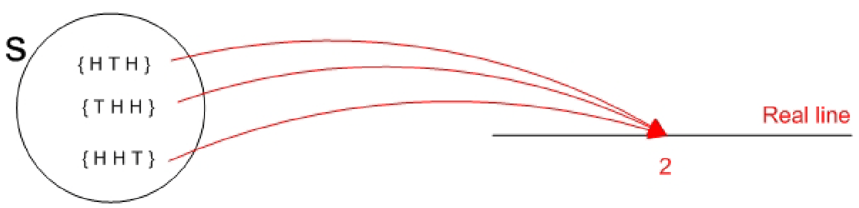
Selecting a number between { 0 and 6 }

$$\{ 0 \leq X \leq 6 \}$$

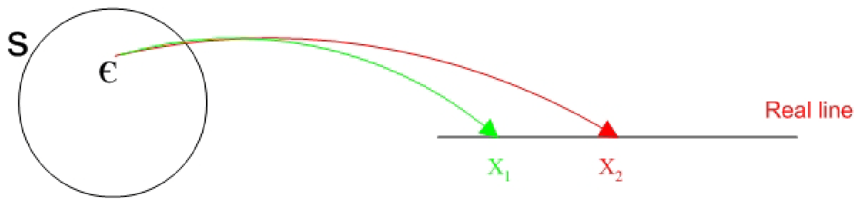
Conditions for a Function to be a Random Variable

The Random variable may be any function that satisfies the followings:

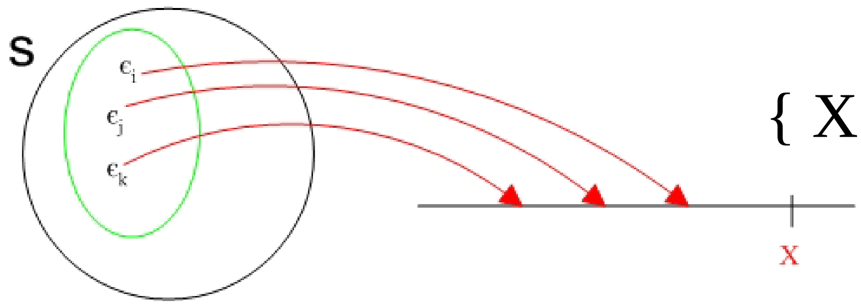
(1) The random variable function can map more than one point in S into same point on the real line.



The random variable function can not be multivalued.



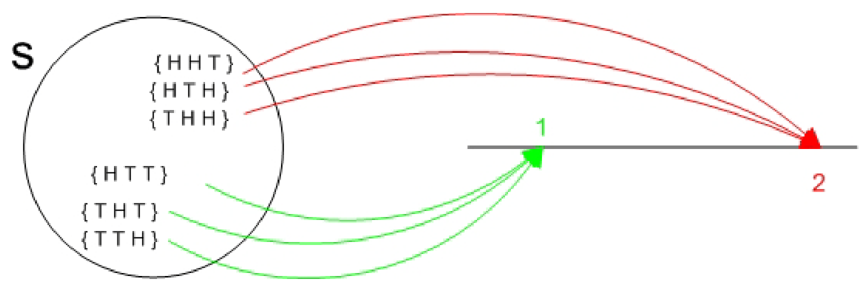
(2) The set $\{ X \le x \}$ corresponds to points in S $\{\epsilon_i \mid X(\epsilon_i) \le x \}$.



$$\{ X \le x \} \overset{\textit{Equivalent}}{\equiv} \{ \varepsilon_i , \varepsilon_j , \varepsilon_k \}$$

$$P\{ X \le x \} \overset{\textit{Equality}}{=} P\{ \varepsilon_i , \varepsilon_j , \varepsilon_k \} = P(\varepsilon_i) + P(\varepsilon_j) + P(\varepsilon_k)$$

Example: Tossing a coin 3 times and observing the number of heads



$$\Rightarrow \{ X \le 1 \} \overset{\textit{Equivalent}}{\equiv} \{ HTT, THT, TTH, TTT \}$$

$$\Rightarrow P\{ X \le 1 \} \overset{\textit{Equality}}{=} P(HTT) + P(THT) + P(TTH) + P(TTT)$$

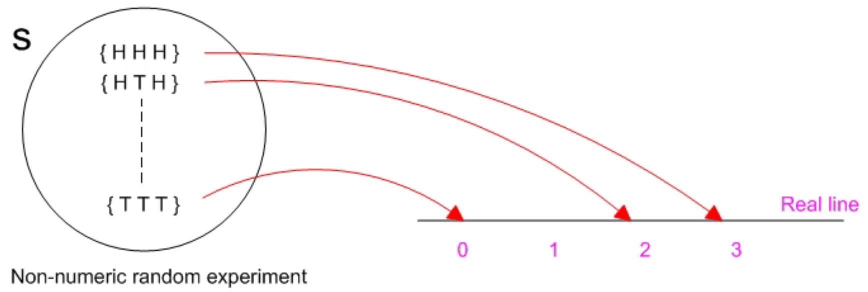
(3) $P\{X=-\infty\}=0 \quad P\{X=\infty\}=0$

This condition does not prevent X from being $-\infty$ or $+\infty$ for some values of S. It only requires that the probability of the set of those S be zero.

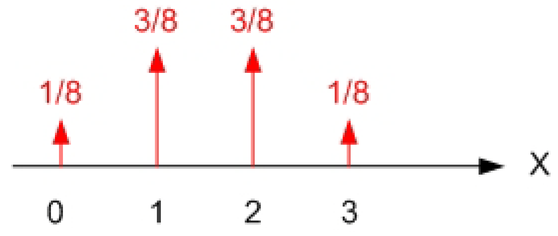
Distribution Function

If you have a random variable X which is numeric by mapping a random experiment outcomes to the real line

Example: Flipping a coin 3 times and counting the number of heads



We can describe the distribution of the R.V X using the **probability**

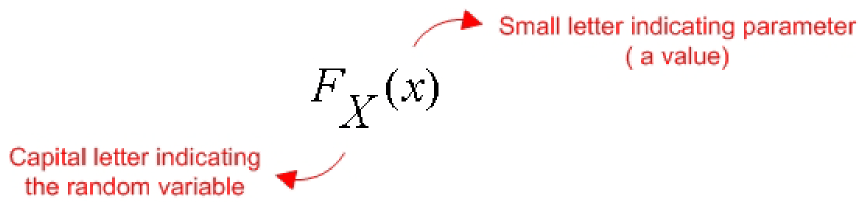


We will define two more distributions of the random variable which will help us finally to calculate probability.

Distribution Function

We define the *cumulative probability distribution function*

$$F_X(x) = P\{X \le x\} \quad \text{where ,}$$



In our flipping the coin 3 times and counting the number of heads

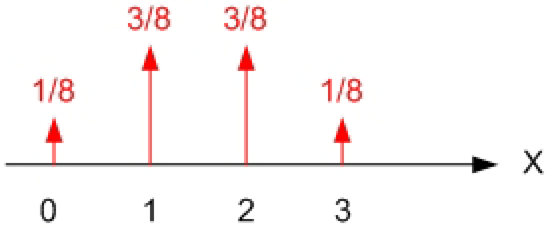
$$F_X(2) = P\{X \le 2\}$$

$$F_X(2) = P\{X \le 2\} = P\{X = 0\} + P\{X = 1\} + P\{X = 2\}$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

Example: Let $X = \{0,1,2,3\}$ with $P(X=0) = P(X=3) = \frac{1}{8}$

$$P(X=1) = P(X=2) = \frac{3}{8}$$

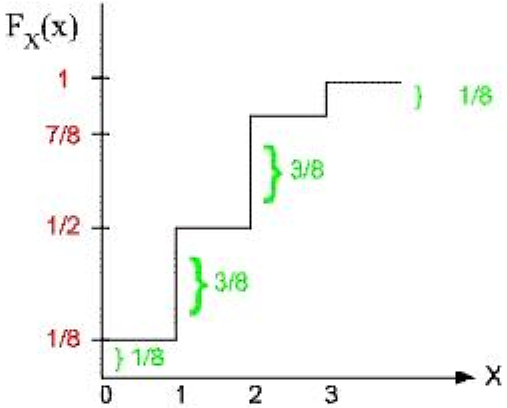


$$F_X(0) = P(X \le 0) = \frac{1}{8}$$

$$F_X(1) = P(X \le 1) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

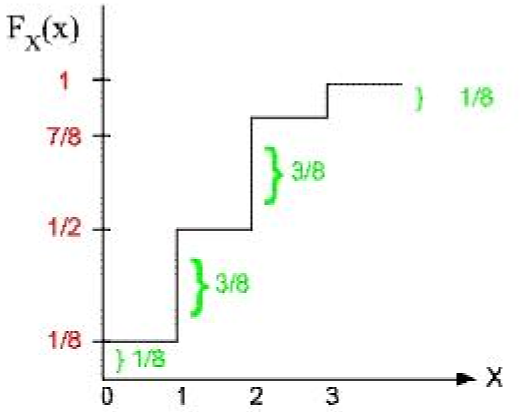
$$F_X(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

$$F_X(3) = P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$



Distribution Function properties

1. $F_X(-\infty) = 0$
2. $F_X(\infty) = 1$
3. $0 \le F_X(x) \le 1$



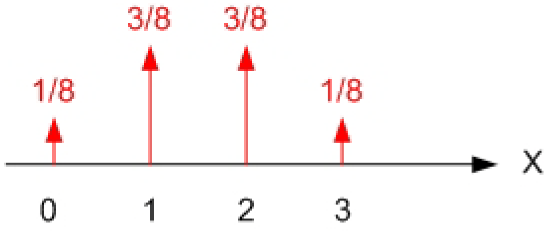
4. $F_X(x_1) \le F_X(x_2) \quad if \; x_1 < x_2$ **Nondecreasing function**

5. $P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$

6. $F_X(x^+) = F_X(x)$ **Continuous from the right**

Example: Let $X = \{0,1,2,3\}$ with $P(X = 0) = P(X = 3) = \frac{1}{8}$

$$P(X = 1) = P(X = 2) = \frac{3}{8}$$

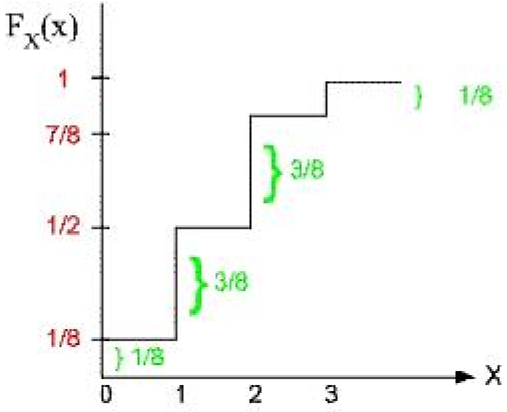


$$F_X(0) = P(X \le 0) = \frac{1}{8}$$

$$F_X(1) = P(X \le 1) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

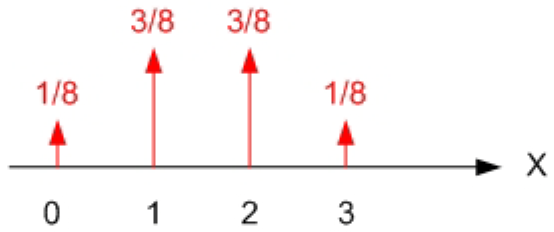
$$F_X(2) = P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$$

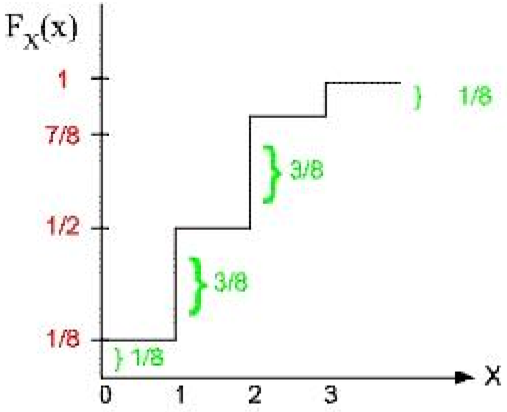
$$F_X(3) = P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$$



Let us consider the experiment of tossing the coin 3 times and observing the number of heads

The probabilities and the distribution function are shown below





The **stair (درج) type** distribution function can be written as

$$F_X(x) = P(X=0)\underbrace{u(x)}_{step\ function} + P(X=1)\underbrace{u(x-1)}_{step\ function} + P(X=2)\underbrace{u(x-2)}_{step\ function}$$

$$+ P(X=3)\underbrace{u(x-3)}_{step\ function}$$

In general $\Rightarrow F_X(x) = \sum_{i=1}^{N} P(X=x_i)u(x-x_i)$ were N can be infinite

Density Function

– We define the derivative of the distribution function $F_X(x)$ as the probability density function $f_X(x)$.

$$f_X(x) = \frac{dF_X(x)}{dx}$$

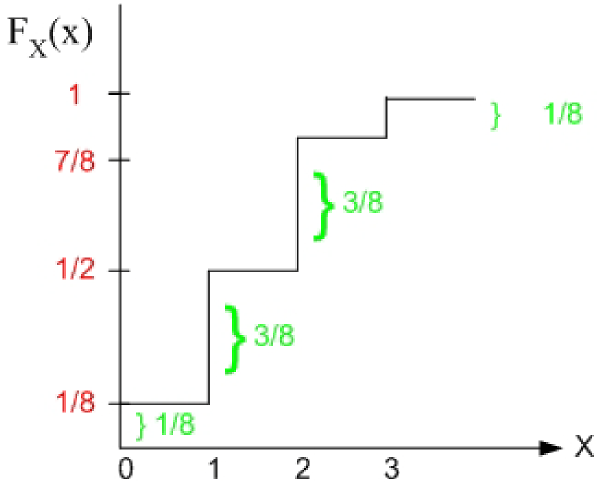
- We call $f_X(x)$ the density function of the R.V X
- In our discrete R.V since

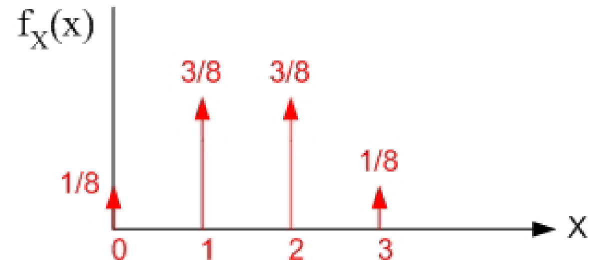
$$F_X(x) = \sum_{i=1}^{N} P(X = x_i) u(x - x_i)$$

$$f_X(x) = \frac{d}{dx}\left(\sum_{i=1}^{N} P(X = x_i) u(x - x_i)\right) = \sum_{i=1}^{N} P(X = x_i) \frac{d}{dx} u(x - x_i)$$

$$= \sum_{i=1}^{N} P(X = x_i) \delta(x - x_i)$$

$$f_X(x) = \sum_{i=1}^{N} P(x_i) \delta(x - x_i)$$





Properties of Density Function

1. $f_X(x) \geq 0$ for all x (Density is non-negative derivative of non-decreasing function)

2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

- Properties (1) and (2) are used to prove if a certain function can be a valid density function.

3. $F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$

From (3)=> $F_X(\infty) = 1$

4. $P\{x_1 < X \leq x_2\} = \int_{x_1}^{x_2} f_X(x)dx$

Since

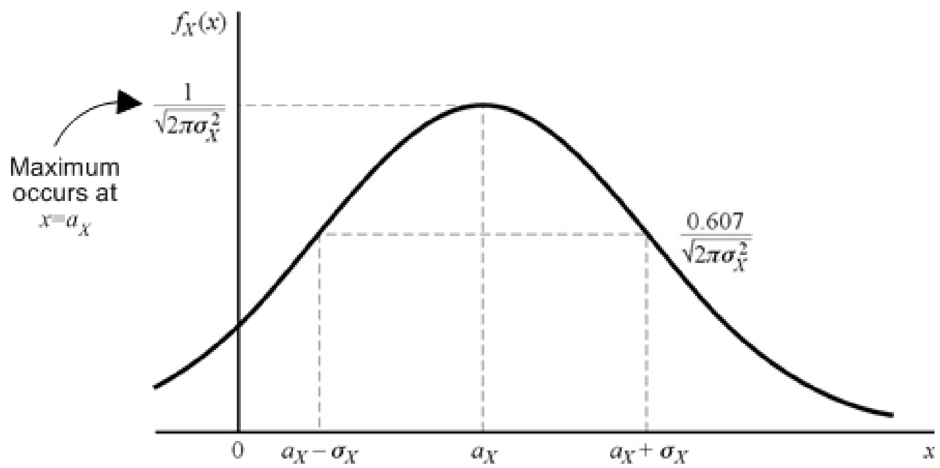
$$P(x_1 < X \leq x_2) = F(x_2) - F(x_1) = \int_{-\infty}^{x_2} f_X(x)dx - \int_{-\infty}^{x_1} f_X(x)dx$$

The Gaussian Random Variable

A random variable X is called Gaussian if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x - a_X)^2/2\sigma_x^2} \quad \textit{where} \quad \sigma_X^2 > 0 \quad \textit{and} \quad -\infty < a_X < \infty$$

are real constants (we will see their significance when we discuss the mean and variance later).



The "spread" about the point $x = a_X$ is related to σ_X

The Gaussian density is the most important of all densities.

It accurately describes many practical and significant real-world quantities such as noise.

The distribution function is found from

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2 / 2\sigma_x^2}\, d\xi$$

The integral has no known closed-form solution and must be evaluated by numerical or approximation method.

However to evaluate numerically for a given x

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2 / 2\sigma_x^2} d\xi$$

We need σ_x^2 and a_x

Example: Let $\sigma_x^2=3$ and $a_x=5$, then

$$F_X(x) = \frac{1}{\sqrt{2\pi 3}} \int_{-\infty}^{x} e^{-(\xi - 5)^2 / 2(3)} d\xi$$

We then can construct the Table for various values of x.

-20	$F_X(-20) = \frac{1}{\sqrt{2\pi 3}} \int_{-\infty}^{-20} e^{-(\xi - 5)^2 / 2(3)} d\xi \Longrightarrow$ Evaluate Numerically
⋮	
+6	$F_X(6) = \frac{1}{\sqrt{2\pi 3}} \int_{-\infty}^{6} e^{-(\xi - 5)^2 / 2(3)} d\xi \Longrightarrow$ Evaluate Numerically
⋮	

Finally we will get a Table for various values of x.

However there is a problem!

The Table will only work for Gaussian distribution with

$\sigma_x^2=3$ and $a_x=5$.

We know that **not all** Gaussian distributions have $\sigma_x^2=3$ and $a_x=5$.

Since the combinations of a_x and σ_x^2 are infinite (uncountable infinite)

$\Longrightarrow$ Uncountable infinite tables to be constructed

$\Longrightarrow$ Unpractical method

We will show that the general distribution function $F_X(x)$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_x^2} d\xi$$

can be found in terms of the normalized Gaussian pdf f(x)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad a_X = 0 \qquad \sigma_X^2 = 1$$

we make the variable change $u = (\xi - a_X)/\sigma_x$ in $F_X(x)$

$$F_X(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x - a_X)/\sigma_x} e^{-u^2/2} du = F(x)$$

$$F_X(x) = F\left(\frac{x - a_X}{\sigma_x}\right)$$

This function F(x) is tabularized in Appendix B for $x \geq 0$

TABLE B-1

Values of $F(x)$ for $0 \leq x \leq 3.89$ in steps of 0.01

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	.9406	.9418	.9429	.9441

For negative value of x we use the relationship $F(-x) = 1-F(x)$

Other Distribution and Density Examples

Binomial

Let $0 < p < 1$, $N = 1, 2, \ldots,$ then the function

$$f_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} \delta(x-k)$$

is called the binomial density function.

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} \text{ is the binomial coefficient}$$

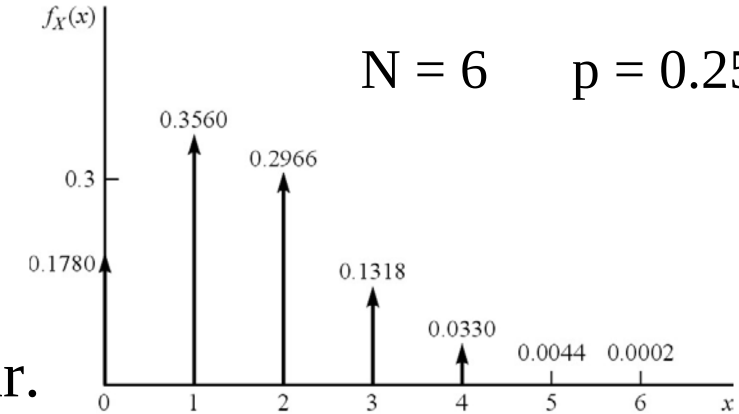
The density can be applied to the

- Bernoulli trial experiment.

- Games of chance.

- Detection problems in radar and sonar.

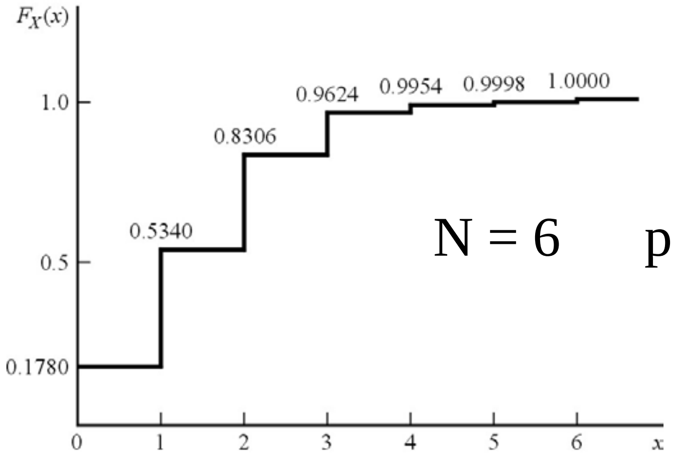
N = 6 p = 0.25



It applies to many experiments that have only two possible outcomes ({H,T}, {0,1}, {Target, No Target}) on any given trial (N).

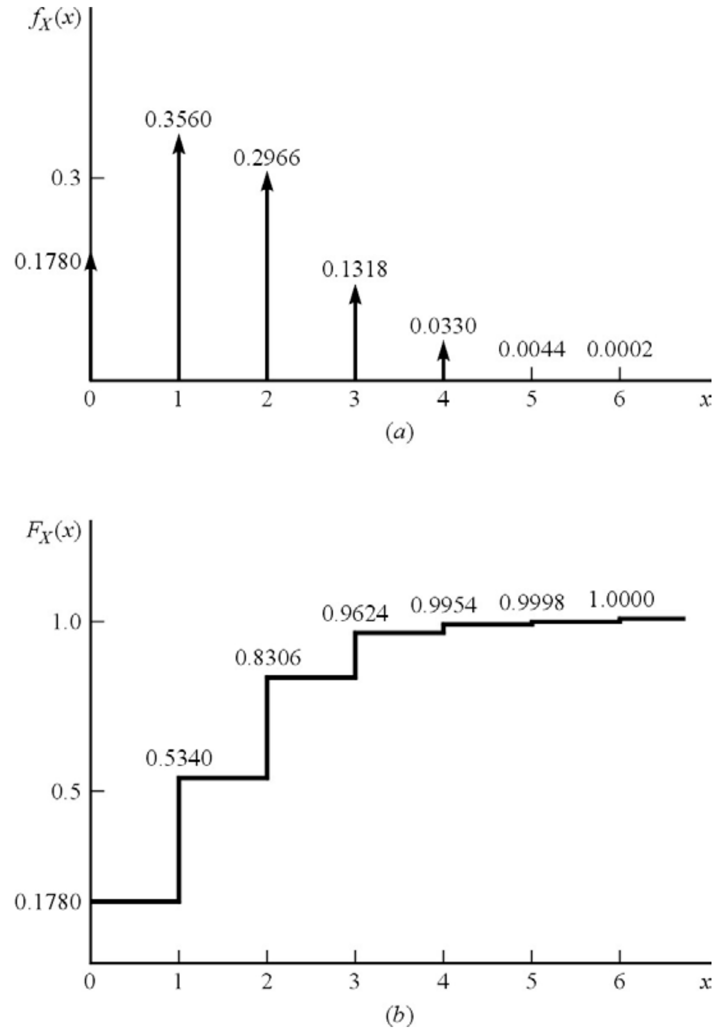
It applies when you have N trials of the experiment of only outcomes and you ask what is the probability of k-successes out of these N trials.

Binomial distribution $F_X(x) = \sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} u(x-k)$



N = 6 p = 0.25

The following figure illustrates the binomial density and distribution functions for N = 6 and p = 0.25.



Poisson

The Poisson RV X has a density and distribution

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k)$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k)$$

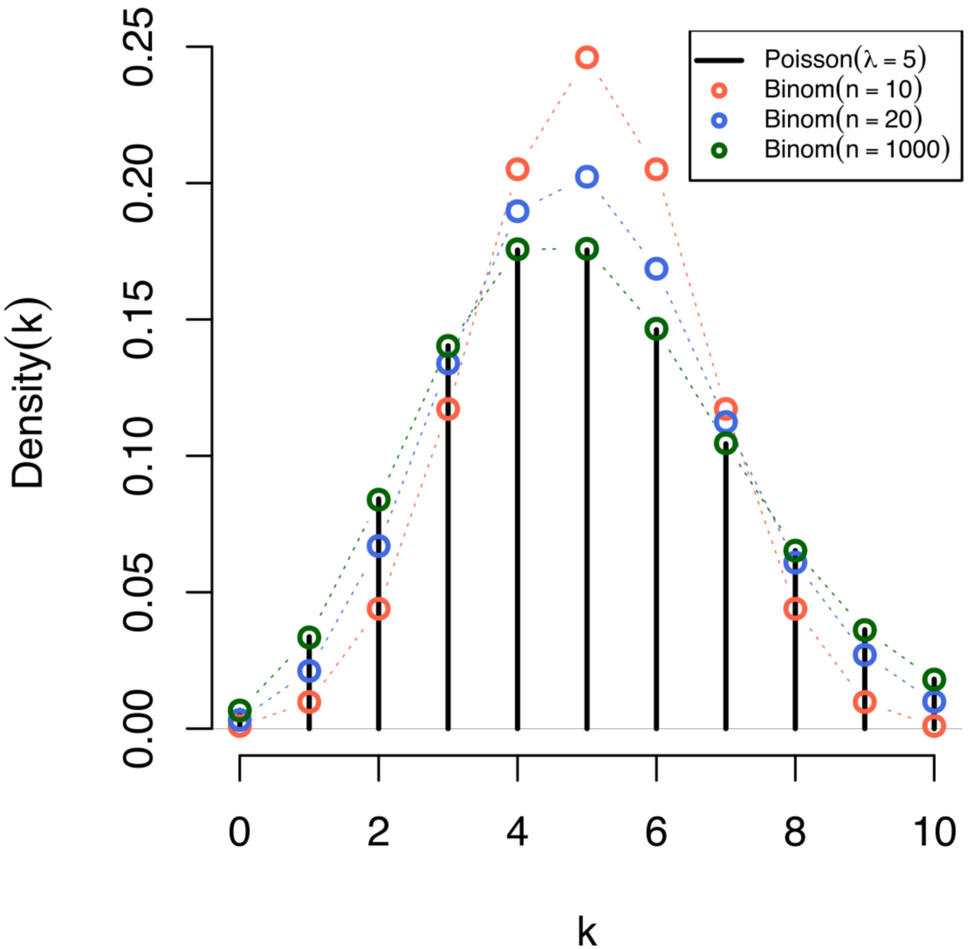
Where $b > 0$ is a real constant.

Binomial → Poisson

$N \rightarrow \infty$

$p \rightarrow 0$

$Np = b$ (constant)



The Poisson RV applies to a wide variety of counting-type applications:

- The number of defective units in a production line.

- The number of telephone calls made during a period of time.

- The number of electrons emitted from a small section of a cathode in a given time interval.

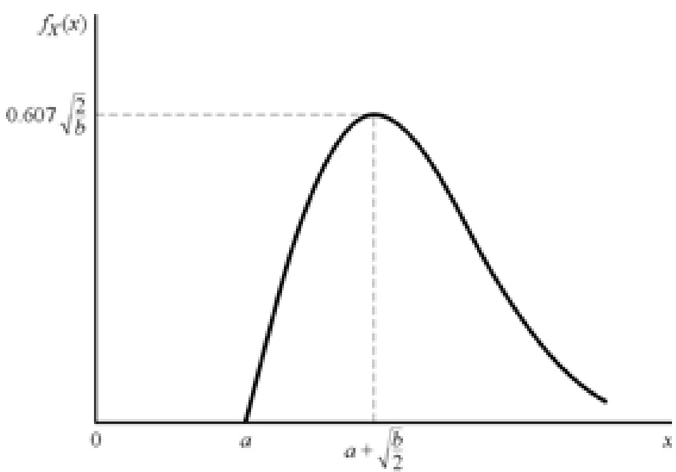
Rayleigh

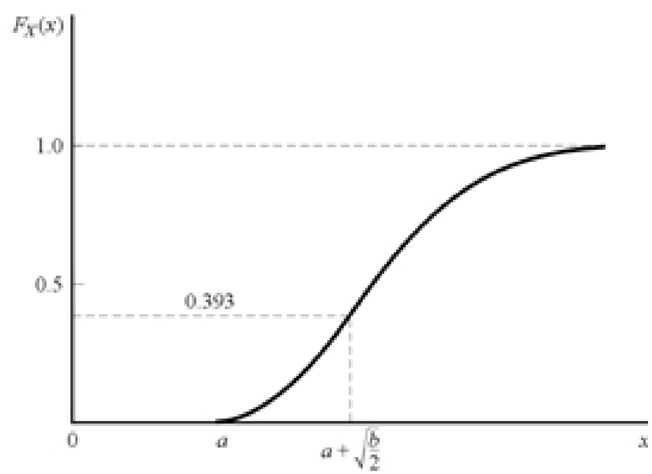
The Rayleigh density and distribution functions are

$$f_X(x) = \begin{cases} \frac{2}{b}(x - a)e^{-(x-a)^2/b} & x \geq a \\ 0 & x < a \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-(x-a)^2/b} & x \geq a \\ 0 & x < a \end{cases}$$

for real constants $-\infty < a < \infty$ and $b > 0$





The Rayleigh density describes some type of noise

Conditional Distribution and Density Functions

For two events A and B the conditional probability of event A given event B had occurred was defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We extend the concept of conditional probability to include random variables

Conditional Distribution

Let X be a random variable and define the event A

$$A = \{X \le x\}$$

we define the conditional distribution function $F_X(x|B)$

$$F_X(x|B) = P\{\overbrace{X \le x}^{A}|B\} = \frac{P\{\overbrace{X \le x \cap B}^{\{X \le x\}\cap B}\}}{P(B)}$$

Properties of Conditional Distribution

(1) $F_X(-\infty|B) = 0$

proof $F_X(-\infty|B) = P\{X \le -\infty|B\}$

$$= \frac{P\{X \le -\infty \cap B\}}{P(B)} = \frac{0}{P(B)} = 0$$

(2) $F_X(\infty|B) = 1$

Pr*oof* $F_X(\infty|B) = P\{X \le \infty|B\}$

$$= \frac{P\{X \le \infty \cap B\}}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(3) $0 \le F_X(x|B) \le 1$

(4) $F_X(x_1|B) \le F_X(x_2|B)$ if $x_1 < x_2$ None Decreasing

(5) $P\{x_1 < X \le x_2|B\} = F_X(x_2|B) - F_X(x_1|B)$

(6) $F_X(x^+|B) = F_X(x|B)$ Right continuous

Conditional Density Functions

We define the Conditional Density Function of the random variable X as the derivative of the conditional distribution function

$$f_X(x|B) = \frac{dF_X(x|B)}{dx}$$

If $F_X(x|B)$ contain step discontinuities as when X is discrete or mixed (continues and discrete) then $f_X(x|B)$ will contain impulse functions.

Properties of Conditional Density

(1) $f_X(x|B) \geq 0$

(2) $\int_{-\infty}^{\infty} f_X(x|B)\,dx = 1$

(3) $F_X(x|B) = \int_{-\infty}^{x} f_X(\xi|B)\,d\xi$

(4) $P\{x_1 < X \leq x_2|B\} = \int_{x_1}^{x_2} f_X(x|B)dx$

Next we define the event $B = \{X \le b\}$ were b is a real number

$$-\infty < b < \infty$$

$$\Rightarrow \quad F_X(x|X \le b) = P\{X \le x|X \le b\} = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}}$$

$$P\{X \le b\} \neq 0$$

Case 1 $\quad x \ge b$

$$\Rightarrow \{X \le b\} \subset \{X \le x\}$$

$$\Rightarrow \{X \le x\} \cap \{X \le b\} = \{X \le b\}$$

$$\Rightarrow F_X(x|X \le b) = \frac{P\{X \le x \cap X \le b\}}{P\{X \le b\}} = \frac{P\{X \le b\}}{P\{X \le b\}} = 1$$

Case 2 $x < b$

$$\Rightarrow \{X \leq x\} \subset \{X \leq b\}$$

$$\Rightarrow \{X \leq x\} \cap \{X \leq b\} = \{X \leq x\}$$

$$\Rightarrow F_X(x|X \leq b) = \frac{P\{X \leq x \cap X \leq b\}}{P\{X \leq b\}} = \frac{P\{X \leq x\}}{P\{X \leq b\}} = \frac{F_X(x)}{F_X(b)}$$

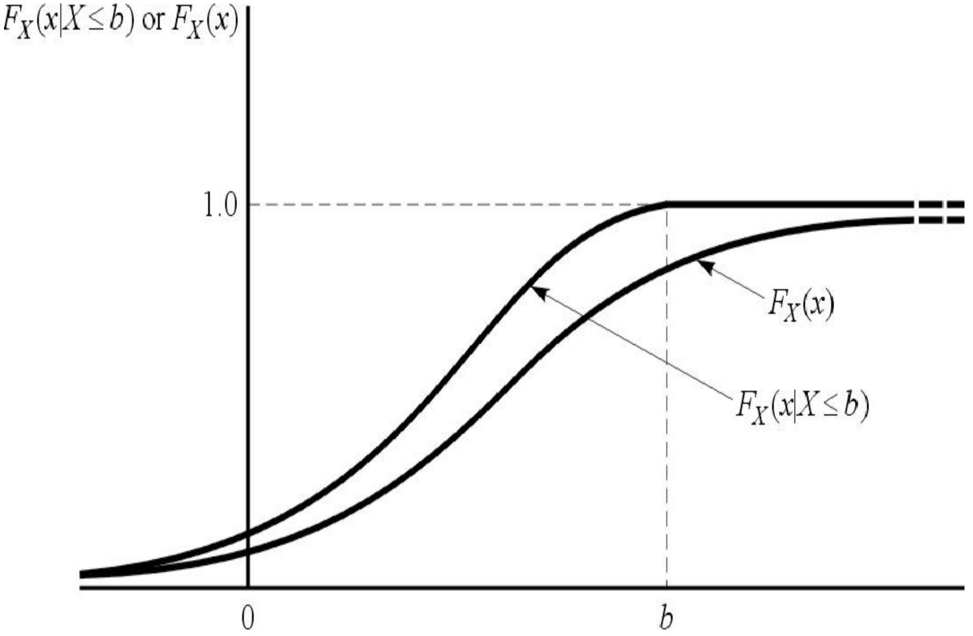
By combining the two expressions we get

$$F_X(x|X \leq b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & x \geq b \end{cases}$$

then the conditional distribution $F_X(x|X \le b)$ is never smaller then the ordinary distribution $F_X(x)$

$$F_X(x|X \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & x \ge b \end{cases}$$

$$\Rightarrow F_X(x|X \le b) \ge F_X(x)$$

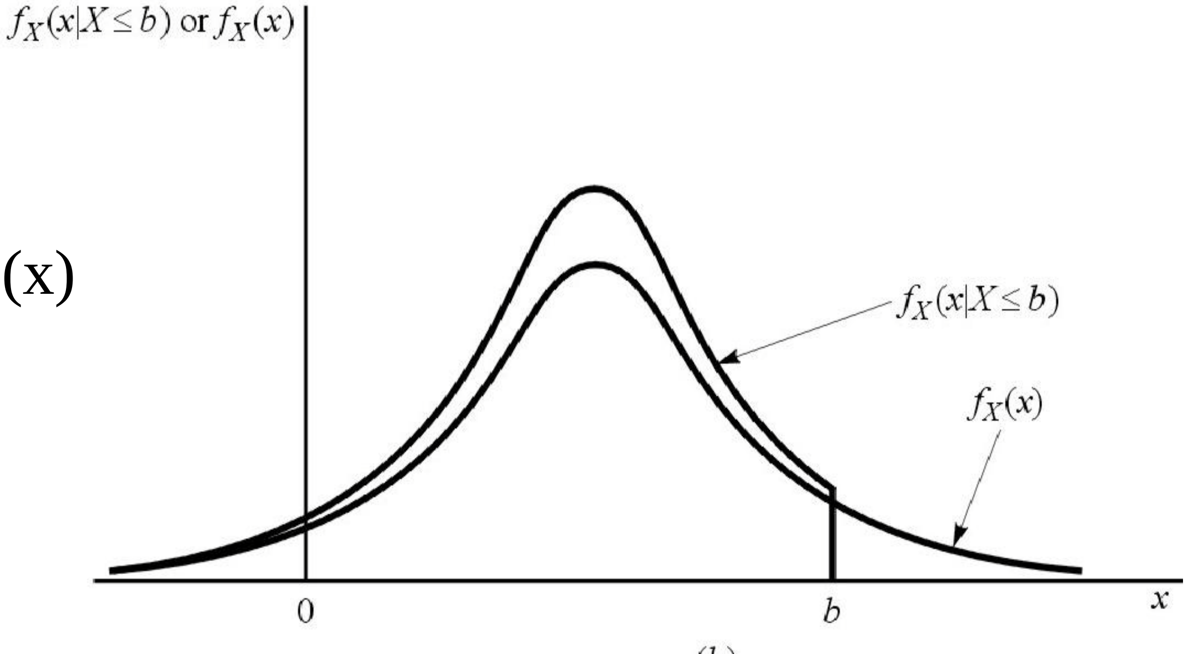


The conditional density function derives from the derivative

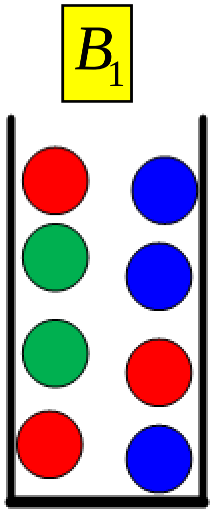
$$f_X(x|X \le b) = \frac{dF_X(x|X \le b)}{dx} = \begin{cases} \dfrac{f_X(x)}{F_X(b)} = \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)dx} & x < b \\ 0 & x \ge b \end{cases}$$

Similarly for the conditional density function

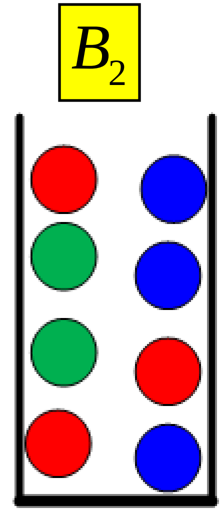
$$\Rightarrow f_X(x|X \le b) \ge f_X(x)$$



Example 2.61-1 Two Boxes have **Red**, **Green** and **Blue** Balls



$$B_1 = \begin{cases} 5 & \text{(red)} \\ 35 & \text{(green)} \\ 60 & \text{(blue)} \end{cases}$$



$$B_2 = \begin{cases} 80 & \text{(red)} \\ 60 & \text{(green)} \\ 10 & \text{(blue)} \end{cases}$$

Our experiment will be to select a box then to select a ball from the box

$P(B_1) = \dfrac{2}{10}$ $\qquad$ $P(B_2) = \dfrac{8}{10}$

B_1 and B_2 are mutually exclusive Events

$P(B_1) + P(B_2) = 1$

Define a discrete random variable X to have values:

$x_1 = 1$ when a **Red** ball is selected

$x_2 = 2$ when a **Green** ball is selected

$x_3 = 3$ when a **Blue** ball is selected

$f(x|B_1)$, $F(x|B_1)$?

$f(x|B_2)$, $F(x|B_2)$?

$f(x)$, $F(x)$?

$P(B_1) = \frac{2}{10}$

$P(B_2) = \frac{8}{10}$

x_i		B_1	B_2	Totals
1	Red	5	80	85
2	Green	35	60	95
3	Blue	60	10	70
		100	150	**250**

$$f(x|B_1) = \sum_{i=1}^{3} P(X = x_i | B_1)\,\delta(x-i)$$

By direct integration

$$F(x|B_1) = \sum_{i=1}^{3} P(X = x_i | B_1)\,u(x-i)$$

$$P(X=1|B_1) = \frac{5}{100} \qquad P(X=2|B_1) = \frac{35}{100} \qquad P(X=3|B_1) = \frac{60}{100}$$

$$f(x|B_1) = \frac{5}{100}\delta(x-1) + \frac{35}{100}\delta(x-2) + \frac{60}{100}\delta(x-3)$$

$$F(x|B_1) = \frac{5}{100}u(x-1) + \frac{35}{100}u(x-2) + \frac{60}{100}u(x-3)$$

Similarly

$P(B_1) = \frac{2}{10}$

$P(B_2) = \frac{8}{10}$

x_i		B_1	B_2	Totals
1	Red	5	80	85
2	Green	35	60	95
3	Blue	60	10	70
		100	150	**250**

$$f\left(x \middle| B_2\right) = \sum_{i=1}^{3} P\left(X = x_i \middle| B_2\right)\delta(x-i)$$

By direct integration

$$F\left(x \middle| B_2\right) = \sum_{i=1}^{3} P\left(X = x_i \middle| B_2\right)u(x-i)$$

$$P\left(X = 1 \middle| B_2\right) = \frac{80}{150} \quad P\left(X = 2 \middle| B_2\right) = \frac{60}{150} \quad P\left(X = 3 \middle| B_2\right) = \frac{10}{150}$$

$$f\left(x \middle| B_2\right) = \frac{80}{150}\delta(x-1) + \frac{60}{150}\delta(x-2) + \frac{10}{150}\delta(x-3)$$

$$F\left(x \middle| B_2\right) = \frac{80}{150}u(x-1) + \frac{60}{150}u(x-2) + \frac{10}{150}u(x-3)$$

x_i		B_1	B_2	Totals
1	Red	5	80	85
2	Green	35	60	95
3	Blue	60	10	70
		100	150	**250**

$$P(B_1) = \frac{2}{10}$$

$$P(B_2) = \frac{8}{10}$$

$$f(x) = \sum_{i=1}^{3} P(X = x_i)\delta(x - i)$$

By direct integration

$$F(x) = \sum_{i=1}^{3} P(X = x_i)u(x - i)$$

$$P(X = 1) = P\left(X = 1 \middle| B_1\right)P(B_1) + P\left(X = 1 \middle| B_2\right)P(B_2)$$

$$= \left(\frac{5}{100}\right)\left(\frac{2}{10}\right) + \left(\frac{80}{150}\right)\left(\frac{8}{10}\right) = 0.437$$

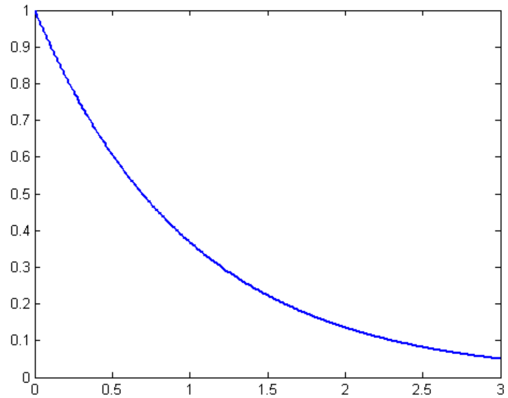
$$P(X = 2) = P\left(X = 2 \middle| B_1\right)P(B_1) + P\left(X = 2 \middle| B_2\right)P(B_2)$$

$$= \left(\frac{35}{100}\right)\left(\frac{2}{10}\right) + \left(\frac{160}{150}\right)\left(\frac{8}{10}\right) = 0.390$$

$$P(X = 3) = P\left(X = 3 \middle| B_1\right)P(B_1) + P\left(X = 3 \middle| B_2\right)P(B_2) = \left(\frac{60}{100}\right)\left(\frac{2}{10}\right) + \left(\frac{10}{150}\right)\left(\frac{8}{10}\right) = 0.173$$

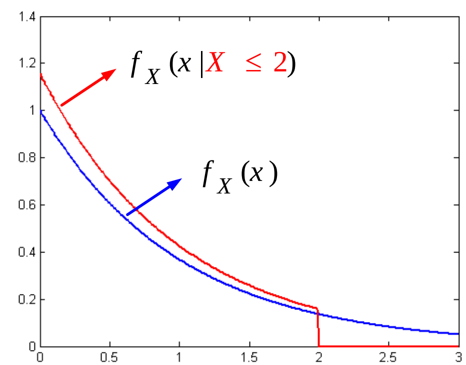
Example 8 Let X be a random variable with an exponential probability density function given as

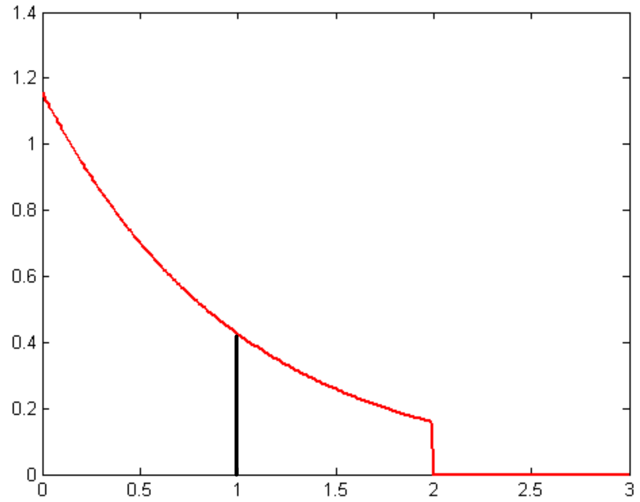
$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Find the probability P(X < 1 | X ≤ 2)

Since $$f_X(x|X \leq 2) = \begin{cases} \dfrac{f_X(x)}{\int_{-\infty}^{2} f_X(x)dx} & x \leq 2 \\ 0 & x > 2 \end{cases} = \begin{cases} \dfrac{e^{-x}}{1-e^{-2}} & x \leq 2 \\ 0 & x > 2 \end{cases}$$





$$P(X < 1 | X \le 2) = \int_0^1 f_X(x | X \le 2)\,dx = \int_0^1 \frac{e^{-x}}{1-e^{-2}}\,dx$$

$$= \frac{\int_0^1 e^{-x}\,dx}{1-e^{-2}} = \frac{1-e^{-1}}{1-e^{-2}} = 0.7310$$