King Fahad University of Petroleum and Minerals

Electrical Engineering

EE315 (Winter 2014)-132

Exam I

24/06/1435

24/4/2014

Time : 6:30-8:00 P.M

NAME	**KEY**
ID	
SER	
SECTION	

Question	Score	Maximum
1		40
2		30
3		30
Total		100

Q1 (40)

Let the probability density function of the random variable X is given by

$$f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $B = \left\{0 < X < \frac{3}{4}\right\}$ be an event and $Y = \text{g}[X] = X^3$ be a transformation on X

(a) Find the mean and variance of X ?

(b) Find $\text{E}[X \mid B]$?

(c) Find the density of Y , $f_Y(y)$?

Solution

(a) $\sigma_X^2 = E[X^2] - \left(E[X]\right)^2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_0^1 x^2(6x(1-x))dx = \frac{3}{10}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 x(6x(1-x))dx = \frac{1}{2}$$

$$\Rightarrow \sigma_X^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}$$

$$(b)\quad E[X \mid B] = \int_{-\infty}^{\infty} x f_{X|B}(x)dx = \int_{0}^{\frac{3}{4}} x f_{X|B}(x)dx$$

$$f_{X|B}(x) = \frac{dF_{X|B}(x)}{dx} = \frac{d}{dx}\begin{cases} \dfrac{F_X(x)}{P(B)} & 0 < x < \dfrac{3}{4} \\ 1 & x > \dfrac{3}{4} \end{cases} = \begin{cases} \dfrac{f_X(x)}{P(B)} & 0 < x < \dfrac{3}{4} \\ 0 & x > \dfrac{3}{4} \end{cases}$$

$$P(B) = \int_{0}^{\frac{3}{4}} f_X(x)dx = \int_{0}^{\frac{3}{4}} 6x(1-x)dx = \frac{27}{32}$$

$$\Rightarrow f_{X|B}(x) = \begin{cases} \dfrac{6x(1-x)}{\frac{27}{32}} & 0 < x < \dfrac{3}{4} \\ 0 & x > \dfrac{3}{4} \end{cases} = \begin{cases} \dfrac{64}{9}x(1-x) & 0 < x < \dfrac{3}{4} \\ 0 & x > \dfrac{3}{4} \end{cases}$$

$$E[X \mid B] = \int_{0}^{\frac{3}{4}} x\left(\frac{64}{9}x(1-x)\right)dx = \frac{7}{16}$$

$$(c)\ f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|_{x=T^{-1}[y]} \qquad Y = T[X] = X^3 \Rightarrow \frac{dy}{dx} = 3x^2 \quad \text{and} \quad x = T^{-1}[y] = \sqrt[3]{y} = (y)^{\frac{1}{3}}$$

$$f_Y(y) = \begin{cases} 6x(1-x)\left[\dfrac{1}{3x^2}\right]\Bigg|_{x=(y)^{\frac{1}{3}}} & 0^{\frac{1}{3}} < x < 1^{\frac{1}{3}} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 6(y)^{\frac{1}{3}}(1-(y)^{\frac{1}{3}})3\left(\dfrac{1}{(y)^{\frac{1}{3}}}\right)^2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2\left[\dfrac{1}{\sqrt[3]{y}} - 1\right] & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Q2 (30)

Let the random variables X and Y have the joint density function

$$f_{XY}(x,y)=\begin{cases}\frac{1}{96}(2x+y) & 0<x<4 \quad 0<y<4\\ 0 & \text{otherwise}\end{cases}$$

(a) Find the densityes $f_X(x)$, $f_Y(y)$?

(b) Are X and Y independents ? *Explain*

(c) Find the the probabilty $P[X-2\le Y\le X]$?

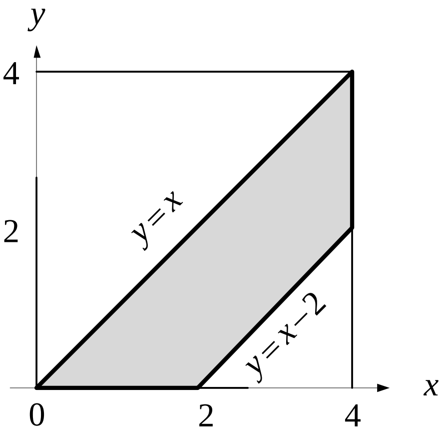
Solution

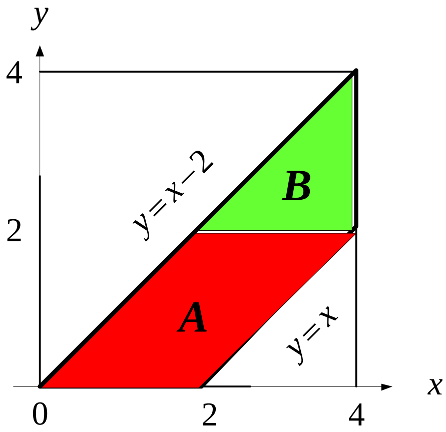
$$(a)\ f_X(x)=\int_{-\infty}^{\infty} f_{XY}(x,y)dy=\int_0^4 \frac{1}{96}(2x+y)dy=\frac{1}{12}x+\frac{1}{12} \qquad 0<x<4$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{XY}(x,y)dx=\int_0^4 \frac{1}{96}(2x+y)dx=\frac{1}{24}y+\frac{1}{6} \qquad 0<y<4$$

(b) Since $f_{XY}(x,y)\ne f_X(x)\,f_Y(y)$ then X and Y are not independents

(c)

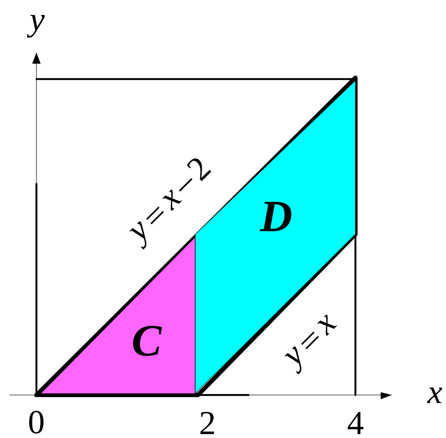




$$P[X-2 \le Y \le X] = \int_{y=0}^{y=2} \int_{x=y}^{x=y+2} \frac{1}{96}(2x+y)dxdy + \int_{y=2}^{y=4} \int_{x=y}^{x=4} \frac{1}{96}(2x+y)dxdy$$

$$= \frac{5}{24} + \frac{7}{36} = \frac{29}{72} = 0.4027$$

OR



$$P[X-2 \le Y \le X] = \int_{0=0}^{x=2} \int_{y=0}^{y=x} \frac{1}{96}(2x+y)dydx + \int_{y=2}^{y=4} \int_{y=x-2}^{y=x} \frac{1}{96}(2x+y)dydx$$

$$= \frac{5}{72} + \frac{1}{3} = \frac{29}{72} = 0.4027$$

Q3 (30)

Let the discrete random variables X and Y have the joint probabilty mass function

$P(x_k, y_j)$	$x_1 = 0$	$x_2 = 1$	$x_3 = 2$
$y_1 = 0$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$y_2 = 1$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
$y_3 = 2$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

(a) Find the covariance $C_{XY}(X,Y)$?

(b) Are X,Y correlated , *explain*?

(c) Are X,Y orthogonal , *explain*?

(d) If $Z = g[X,Y] = X + Y$, find the probability density of Z , $f_Z(z)$?

(write a mathematical expression)

Solution

(a) $C_{XY}(X,Y) = R_{XY} - E[X]E[Y] = E[XY] - E[X]E[Y]$

$$E[XY] = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j P(X = x_i, Y = y_j)$$

$$= (0)(0)\left(\frac{1}{9}\right) + (0)(1)\left(\frac{1}{9}\right) + (0)(2)\left(\frac{1}{9}\right)$$

$$+ (1)(0)\left(\frac{1}{9}\right) + (1)(1)\left(\frac{1}{9}\right) + (1)(2)\left(\frac{1}{9}\right)$$

$$+ (2)(0)\left(\frac{1}{9}\right) + (2)(1)\left(\frac{1}{9}\right) + (2)(2)\left(\frac{1}{9}\right) = \frac{9}{9} = 1$$

$$E[X] = \sum_{i=1}^{3} x_i P(X = x_i)$$

$$P(X = x_i) = \sum_{j=1}^{3} P(X = x_i, Y = y_j)$$

$$\Rightarrow \; P(X = 0) = \sum_{j=1}^{3} P(X = 0, Y = y_j)$$

$$= P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2)$$

$$= \left(\frac{1}{9}\right) + \left(\frac{1}{9}\right) + \left(\frac{1}{9}\right) = \frac{1}{3}$$

Similarly

$$P(X = 1) \; = \sum_{j=1}^{3} P(X = 1, Y = y_j) = \frac{1}{3}$$

$$P(X = 2) = \sum_{j=1}^{3} P(X = 2, Y = y_j) = \frac{1}{3}$$

$$\Rightarrow E[X] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right) = 1$$

$$E[Y] = \sum_{j=1}^{3} y_i P(Y = y_j)$$

$$P(Y = y_j) = \sum_{i=1}^{3} P(X = x_i, Y = y_j)$$

$$\Rightarrow \; P(Y = 0) = \sum_{i=1}^{3} P(X = x_i, Y = 0)$$

$$= P(X = 0, Y = 0) + P(X = 1, Y = 0) + P(X = 2, Y = 0)$$

$$= \left(\frac{1}{9}\right) + \left(\frac{1}{9}\right) + \left(\frac{1}{9}\right) = \frac{1}{3}$$

Similarly

$$P(Y = 1) \; = \sum_{i=1}^{3} P(X = x_i, Y = 1) = \frac{1}{3}$$

$$P(X = 2) = \sum_{j=1}^{3} P(X = x_i, Y = 2) = \frac{1}{3}$$

$$\Rightarrow E[Y] = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{3}\right) = 1$$

$$\Rightarrow \mathrm{C}_{XY}\left(X, Y\right) = E[XY] - E[X]E[Y] = 1 - (1)(1) = 0$$

(b) Since Cov(X,Y) = 0 $\Rightarrow$ X, Y are uncorrelated

(c) Since $E[XY] = 1 \neq 0$ $\Rightarrow$ X, Y are not orthognal

(d) $Z = g[X,Y] = X + Y \quad \Rightarrow \quad Z = 0,1,2,3,4$

$$\{Z=0\} \equiv \{(0,0)\} \Rightarrow P(Z=0) = P(0,0) = \frac{1}{9}$$

$$\{Z=1\} \equiv \{(0,1) \cup (1,0)\} \Rightarrow P(Z=1) = P(0,1) + P(1,0) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$\{Z=2\} \equiv \{(0,2) \cup (1,1) \cup (2,0)\} \Rightarrow P(Z=2) = P(0,2) + P(1,1) + P(2,0) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$$

$$\{Z=3\} \equiv \{(1,2) \cup (2,1)\} \Rightarrow P(Z=3) = P(0,1) + P(1,0) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$\{Z=4\} \equiv \{(2,2)\} \Rightarrow P(Z=4) = P(2,2) = \frac{1}{9}$$

$$f_Z(z) = \frac{1}{9}\delta(z) + \frac{2}{9}\delta(z-1) + \frac{1}{3}\delta(z-2) + \frac{2}{9}\delta(z-3) + \frac{1}{9}\delta(z-4)$$