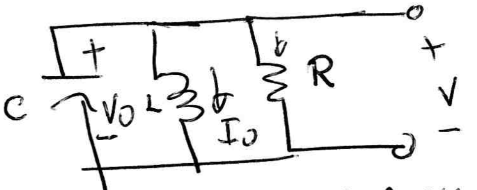
In this chapter we extend the natural response and step response to circuits containing both indictors and capacitors and resistors.

We will consider two simple structures:

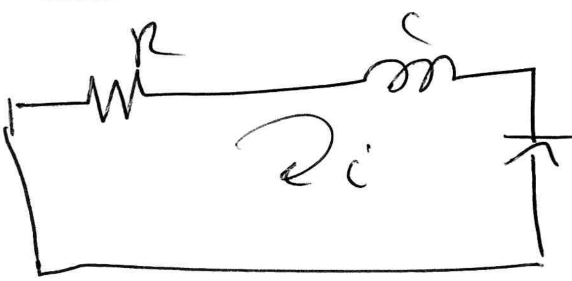
The parallel RLC



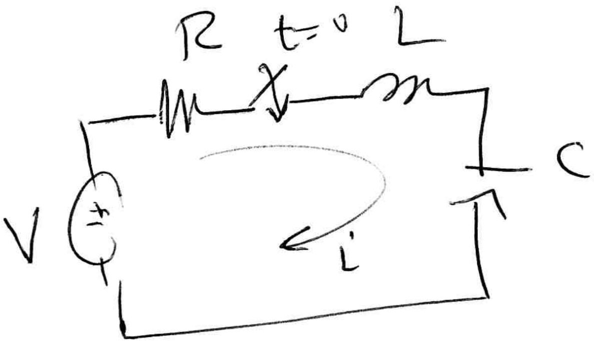
Natural response

step resp...

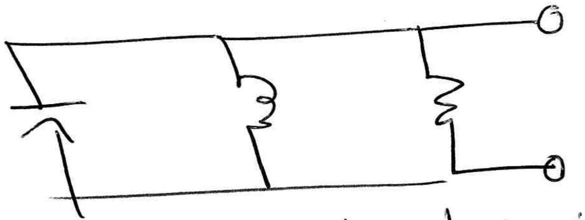
The seires RLC



natural



8.10 The natural response of parallel RLC



KCL $\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + I_0 + C\frac{dv}{dt} = 0$

eliminat the integral ⇒ $\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{dv^2}{dt^2} = 0$

Normalize the equation by dividing by C

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

This is a 2^{ed} order differntial equation
(highest order derivative is 2)

Constant coefficients $\frac{1}{RC}$ $\frac{1}{LC}$

In this chapter we call ~~such~~ circuits were
the highest order of derivative is 2
second order circuits

The general solution of the seond-order-
differential equation

In equation $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$

we can not solve it by separting the
variables and integrating as we were able
to do with 1^{st} order equations in chapter 7.

The classical approach to solve the 2^{ed} order
equation is to assume that the solution
is of exponential form

$$v(t) = A e^{st}$$

were A and s are unknown Constants

Q: what type of a function that will satisfied the 2nd order differntial equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

A: We are seeking a function such that when we add to its 2nd derivative $\left(\frac{d^2v}{dt}\right)$ a constant multiplied by the 1st derivation $\left(\frac{1}{RC}\frac{dv}{dt}\right)$

and also add to it a constant multiplied by the function $\left(\frac{1}{LC}v\right)$

we get zero

The only function that satisfies this is of exponantial nature similar to the one that satisfies the 1st order differential equation

$$v(t) = Ae^{st}$$

were A and s are unknown constants

Now we substitute the proposed solution

$$\cancel{x}\ v(t) = A e^{-st}$$

into the 2ed order differntial equation

$$\frac{d^2 v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{v(t)}{LC} = 0$$

we get

$$A s^2 e^{st} + \frac{As}{RC} e^{st} + \frac{A e^{st}}{LC} = 0$$

simplfying

$$A e^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

a solution to this is

$A = 0$ which can not be since that will make the solution $v(t) = A e^{st} = 0 \ \forall t$ which is a physical impossible since energy is stored in either the inductor or capacitor

$e^{st} = 0$ which can not for any finite value of st

That will leave the only solution

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

This equation is called the "characteristic equation" of the differential equation

The roots of ~~x~~ this quadratic equation determine the mathematical character of $v(t)$, this why it is called the characteristic equation.

The two roots of the characteristic ~~eq~~ equation are

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

If either root s_1 or s_2 substitute into the proposed solution $v(t) = Ae^{st}$ ~~it~~ will satisfies the given differential equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

This result holds regardless of the value of A

$$A e^{st} \underbrace{\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right)}_{\text{this will equal zero for } s_1 \text{ or } s_2 \text{ regardless of the value of } A} = 0$$

$\Rightarrow$ Will satisfy the solution $v(t) = Ae^{st}$ regardless of A

Therefore,

$$v_1(t) = A_1 e^{s_1 t} \quad \text{and} \quad v_1(t) = A_2 e^{s_2 t}$$

will satisfies the differntial equation

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

i.e $v_1(t) = A_1 e^{s_1 t}$

$$\Rightarrow \underbrace{A_1 e^{s_1 t} \underbrace{\left(s_1^2 + \frac{s_1}{RC} + \frac{1}{LC}\right)}_{=0}}_{=0} = 0$$

$$\Rightarrow v_1(t) = A_1 e^{s_1 t} \text{ is a solution}$$

Similarly $v_2(t) = A_2 e^{s_2 t}$

$$\Rightarrow \underbrace{A_2 e^{s_2 t} \underbrace{\left(s_2^2 + \frac{s_2}{RC} + \frac{1}{LC}\right)}_{0}}_{=0} = 0$$

$$\Rightarrow v_2(t) = A_2 e^{s_2 t} \text{ is a solution.}$$

~~We also~~ ~~Also v(t) = v₁(t) + v₂(t) = A₁e^{s₁t} + A₂e^{s₂t}~~

is a solution

If

Also the sum of the two solutions,

$$v(t) = v_1(t) + v_2(t) \quad \text{is a solution}$$
$$= A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

substituting $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ into the differential equation ~~we get~~ $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$, we get.

$$\frac{dv(t)}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$\frac{d^2 v(t)}{dt} = A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t}$$

substituting into the differntial equation, we get

$$\underbrace{A_1 e^{s_1 t} \underbrace{\left(s_1^2 + \frac{1}{RC} s_1 + \frac{1}{LC} \right)}_{=0}}_{=0} + \underbrace{A_2 e^{st} \underbrace{\left(s_2^2 + \frac{1}{RC} s_2 + \frac{1}{LC}\right)}_{=0}}_{=0} = 0$$

$\Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ is a solution to the differntial equation

$\Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ is the natural response of the parallel RLC

Therefore,

$$v_1(t) = A_1 e^{s_1 t} \quad \text{is a solution}$$

$$v_2(t) = A_2 e^{s_2 t} \quad \text{" "}$$

$$v(t) = v_1(t) + v_2(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{is a solution}$$

However $v(t) = v_1(t) + v_2(t)$ is the general solution

The roots of the characteristic equation $s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{L}{LC}}$$

are determined by the circuit parameters R, L and C

the initial conditions I_0 ~~and V_0~~ (current on the inductor) and V_0 (voltage across the capacitor) determine the the constants A_1 and A_2.

The behavior of the solution $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ depends on the value of s_1 and s_2

9

We write s_1 and s_2 as

$$s_{1,2} = -\underbrace{\frac{1}{2RC}}_{\alpha} \pm \sqrt{\underbrace{\left(\frac{1}{2RC}\right)^2}_{\alpha^2} - \underbrace{\frac{1}{LC}}_{\omega_0^2}}$$

were, $\alpha = \frac{1}{2RC}$ $\omega_0 = \frac{1}{\sqrt{LC}}$

$$\Rightarrow \quad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Since, $v_1(t) = e^{\overbrace{s_1 t}^{\text{dimensionless}}}$ $\Rightarrow$ $v_2(t) = e^{\overbrace{s_2 t}^{\text{dimensionless}}}$,

$\Rightarrow$ s_1 and s_2 have the dimension of the reciprocal of time or frequency

$\Rightarrow$ α and ω_0 also have dimension of ~~to~~ reciprocal of time ~~or~~ or frequency

To distinguish among the frequencies s_1, s_2, α and ω_0

s_1 and s refered to as Complex frequencies

Note s_1, s_2 can be complex

α neper frequency

ω_0 resonant frequency.

we will see the significant of ~~these~~ this terminology as we move through the Chaptar

All these frequencies have the dimension of angular frequency per time ~~unit~~ or radian/second (rad/s).

The nature of the roots s_1 and s_2 depend on the values of α and w_0 or LRC.

There are three possible outcomes

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

1st If $w_0^2 < \alpha^2 \Rightarrow$ both ~~roots~~ s_1 and s_2 will be real and distinct

The voltage responce is said to be overdamped as will be discussed later

2ed If $w_0^2 > \alpha^2 \Rightarrow$ both s_1 and s_2 will be complex conjugate

The voltage responce is saide to be underdamped

3ed If $\omega_0^2 = \alpha^2 \Rightarrow$ s_1 and s_2 will be equal and real

The voltage responce is said to be critically damped

~~To fin~~

$\alpha = \frac{1}{2RC}$ Can be found from the circuit parameter R and C

The Constants D_1 and D_2 Can be found as follows:

$$v(0^+) = V_0 = D_2 \quad — ①$$

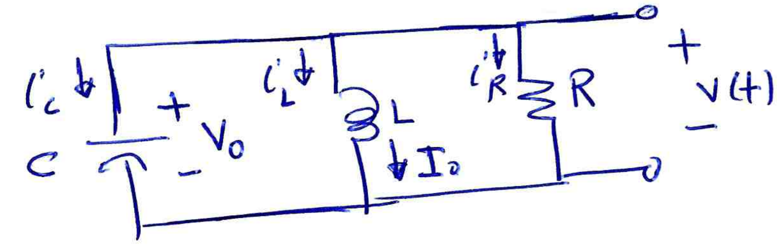
$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = D_1 - \alpha D_2 \quad — ②$$

In practice, it is rarely we encounter critical damp because that require

$$\alpha = \frac{1}{2RC} = w_0 = \frac{1}{\sqrt{LC}}$$

which is difficult to select real Components RLC that will satisfies $\alpha = w_0$.

8.2 The Forms of the Natural Response of a Parallel RLC Circuit



By KCL

$$\frac{d^2 v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{v(t)}{LC} = 0$$

2ed order diff. eqn.

$$v(t) = v_1(t) + v_2(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$S_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha = \frac{1}{2RC}$ Neper freq

$\omega_0 = \frac{1}{\sqrt{LC}}$ resonant freq

S_1, S_2 Complex freq.

(1) If $\alpha^2 > \omega_0^2$ S_1, S_2 are real and distinct

the response $v(t)$ is overdamped.

(2) If $\alpha^2 < \omega_0^2$ S_1, S_2 are Complex Conjugate

the response $v(t)$ is underdamped

(3) If $\alpha^2 = \omega_0^2$ $S_1 = S_2 = -\alpha = -\frac{1}{2RC}$

the voltage response $v(t)$ is critically damped.

$$v(t) = A_1 e^{s_1 t} + \cancel{A_2 e^{s_2 t}} A_2 e^{s_2 t}$$

s_1 and s_2 are determined as

$$\cancel{s_{1,2} = -\alpha \pm}$$

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

s_1, s_2 Can be determined from RLC

A_1, A_2 Can be determined from matching the $v(t)$ to the initial Conditions $v(0)$, $i(0)$

we will analyze the natural response for the three type of damping.

(I) The overdamped Voltage Response

The roots s_1, s_2 are real and distinct $(\alpha^2 > \omega_0^2)$

~~The~~ A_1, A_2 are determined from the initial Conditions as follows:

$$v(0^+) = V_0 = A_1 + A_2 \quad — (1)$$

we need a 2ed question ~~reat~~ relating A_1 and A_2

$$\frac{dv(t)}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2$$

Know how to find $\frac{dv(0^+)}{dt}$

Since,

$$i_c(t) = c\,\frac{dv(t)}{dt}$$

$$\Rightarrow \quad i_c(0^+) = c\,\frac{dv(0^+)}{dt}$$

$$\Rightarrow \quad \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

From the circuit, by kcl,

$$i_c(t) + i_L(t) + i_R(t) = 0$$

$$\Rightarrow \quad i_c(0^+) + i_L(0^+) + i_R(0^+) = 0$$

$$i_L(0^+) = I_0 \quad \text{(The initial Condition).}$$

$$i_R(0^+) = \frac{v(0^+)}{R} = \frac{V_0}{R}$$

$$\Rightarrow \quad i_c(0^+) = -\frac{V_0}{R} - I_0$$

$$\Rightarrow \quad \frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{c} = -\frac{V_0}{RC} - \frac{I_0}{C} \; \cancel{= S_1 A_1 +} S_2 A$$

$$\Rightarrow \quad S_1 A_1 + S_2 A_2 = -\frac{V_0}{RC} - \frac{I_0}{C} \quad \text{— (2)}$$

The 2ed equation needed

note: s_1 and s_2 are determined from RLC

~~Sol~~

solving (1) and (2) we find A_1, A_2

$$\Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Example 8.2

Example 8.2 Finding the Overdamped Natural Response of a Parallel *RLC* Circuit

For the circuit in Fig. 8.6, $v(0^+) = 12$ V, and $i_L(0^+) = 30$ mA.

a) Find the initial current in each branch of the circuit.

b) Find the initial value of dv/dt.

c) Find the expression for $v(t)$.

d) Sketch $v(t)$ in the interval $0 \le t \le 250$ ms.

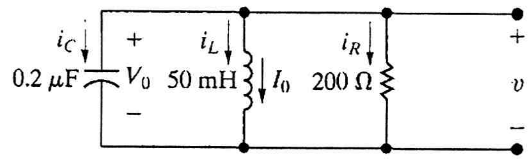


Figure 8.6 ▲ The circuit for Example 8.2.

Solution

a) The inductor prevents an instantaneous change in its current, so the initial value of the inductor current is 30 mA:

$$i_L(0^-) = i_L(0) = i_L(0^+) = 30 \text{ mA}.$$

The capacitor holds the initial voltage across the parallel elements to 12 V. Thus the initial current in the resistive branch, $i_R(0^+)$, is 12/200, or 60 mA. Kirchhoff's current law requires the sum of the currents leaving the top node to equal zero at every instant. Hence

$$i_C(0^+) = -i_L(0^+) - i_R(0^+)$$
$$= -90 \text{ mA}.$$

Note that if we assumed the inductor current and capacitor voltage had reached their dc values at the instant that energy begins to be released, $i_C(0^-) = 0$. In other words, there is an instantaneous change in the capacitor current at $t = 0$.

Capacitor current can jump

b) Because $i_C = C(dv/dt)$,

$$\frac{dv(0^+)}{dt} = \frac{-90 \times 10^{-3}}{0.2 \times 10^{-6}} = -450 \text{ kV/s}.$$

c) The roots of the characteristic equation come from the values of R, L, and C. For the values specified and from Eqs. 8.14 and 8.15 along with 8.16 and 8.17,

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$
$$= -12{,}500 + 7500 = -5000 \text{ rad/s},$$
$$s_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8}$$
$$= -12{,}500 - 7500 = -20{,}000 \text{ rad/s}.$$

Because the roots are real and distinct, we know that the response is overdamped and hence has the form of Eq. 8.18. We find the co-efficients A_1 and A_2 from Eqs. 8.23 and 8.24. We've already determined s_1, s_2, $v(0^+)$, and $dv(0^+)/dt$, so

$$12 = A_1 + A_2,$$

$$-450 \times 10^3 = -5000A_1 - 20{,}000A_2.$$

We solve two equations for A_1 and A_2 to obtain $A_1 = -14$ V and $A_2 = 26$ V. Substituting these values into Eq. 8.18 yields the overdamped voltage response:

$$v(t) = (-14e^{-5000t} + 26e^{-20{,}000t}) \text{ V}, \quad t \ge 0.$$

As a check on these calculations, we note that the solution yields $v(0) = 12$ V and $dv(0^+)/dt = -450{,}000$ V/s.

d) Figure 8.7 shows a plot of $v(t)$ versus t over the interval $0 \le t \le 250$ ms.

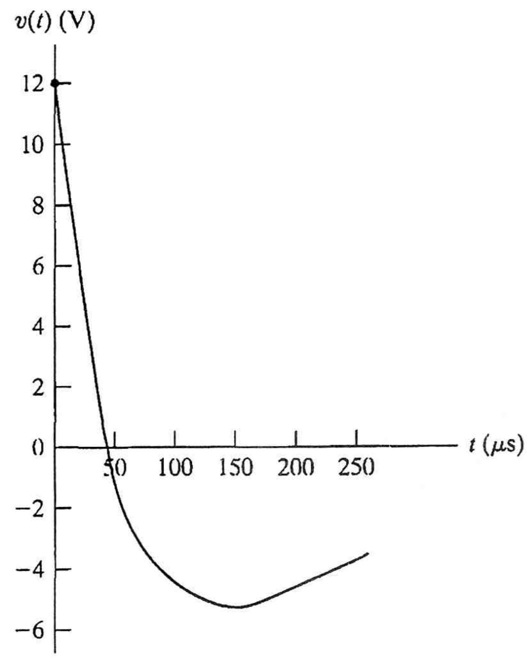


Figure 8.7 ▲ The voltage response for Example 8.2.

(II) The underdamped Voltage Response

$w_0^2 > \alpha^2 \Rightarrow s_1$ and s_2 are Complex Conjugate

$$s_1 = -\alpha + \sqrt{\alpha^2 - w_0^2}$$
$$= -\alpha + \sqrt{-(w_0^2 - \alpha^2)}$$
$$= -\alpha + j\sqrt{w_0^2 - \alpha^2}$$
$$= -\alpha + j w_d$$

were $w_d = \sqrt{w_0^2 - \alpha^2}$ is a frequency called " damped radian frequency"

similarly $s_2 = -\alpha - j w_d$

$$\Rightarrow v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$
$$= A_1 e^{-(\alpha - j w_d)t} + A_2 e^{-(\alpha + j w_d)t}$$
$$= e^{-\alpha t}\left(A_1 e^{j w_d t} + A_2 e^{-j w_d t}\right)$$

Using Euler identies: $e^{j\theta} = \cos\theta + j\sin\theta$, $e^{-j\theta} = \cos\theta - j\sin\theta$

$$\Rightarrow v(t) = e^{-\alpha t}\left[A_1(\cos w_d t + j\sin w_d t) + A_2(\cos w_d t - j\sin w_d t)\right]$$
$$= e^{-\alpha t}\left[(A_1 + A_2)\cos w_d t + j(A_1 - A_2)\sin w_d t\right]$$

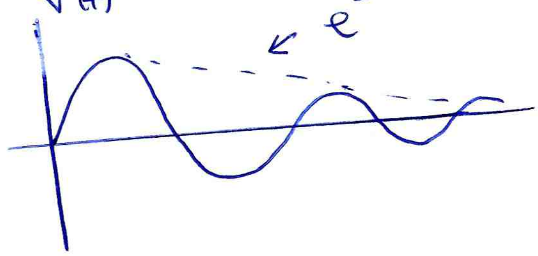
Let,

$$B_1 = (A_1 + A_2)$$

$$B_2 = j(A_1 - A_2)$$

$$\Rightarrow v(t) = e^{-\alpha t}[B_1 \cos w_d t + B_2 \sin w_d t]$$

It is clear that the natural response for this case is exponentially damped and oscillatory in nature



The Constants B_1 and B_2 are real because $v(t)$ is real (the left hand side) therefore the right hand side is also real

In the ~~undeda~~ underdamped Case A_1 and A_2 are Complex Conjugate (Can be shown) P. 8.12, 8.13

~~Therefore $A_1 + A_2$ is real~~

example

Therefore $A_1 + A_2 = A_1 + A_1^* = 2\,Re(A_1)$

$A_1 - A_2 = A_1 - A_1^* = j2\,Im(A_1)$

$\Rightarrow B_1 = A_1 + A_2$ is ~~real~~ $= 2\,Re(A_1)$

$B_2 = j(A_1 - A_2) = -2\,Im(A_1)$

$$v(t) = B_1 e^{-\alpha t} \cos w_d t + B_2 e^{-\alpha t} \sin w_d t$$

$$\alpha = \frac{1}{2RC} \qquad w_d = \sqrt{w_0^2 - \alpha^2} = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$$

Can be determined from RLC

B_1 and B_2 Can be determined from initial Conditions $v(0^+) = V_0$ and $i(0^+) = I_0$ as follows:

$$v(0^+) = V_0 = B_1 e^{-0} \cos(0) + B_2 e^{-0} \sin(0) = B_1 \quad -①$$

$$\frac{dv(t)}{dt} = -\alpha B_1 e^{-\alpha t} \cos w_d t - w_d B_1 e^{-\alpha t} \sin w_d t - \alpha B_2 e^{-\alpha t} \sin w_d t + w_d B_2 e^{-\alpha t} \cos w_d t$$

$$\frac{dv(t)}{dt} = -\alpha B_1 - 0 - 0 + w_d B_2 = -\alpha B_1 + w_d B_2$$

It was shown that $\frac{dv(0^+)}{dt} = -\frac{V_0}{RC} - \frac{I_0}{C}$

$$\Rightarrow \quad -\alpha B_1 + w_d B_2 = -\frac{V_0}{RC} - \frac{I_0}{C} \quad -②$$

solving ① , ② we obtain B_1 and B_2

Ex 8.4

Example 8.4 Finding the Underdamped Natural Response of a Parallel *RLC* Circuit

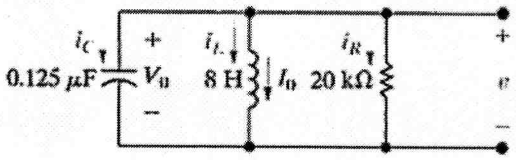
In the circuit shown in Fig. 8.8, $V_0 = 0$, and $I_0 = -12.25$ mA.

a) Calculate the roots of the characteristic equation.

b) Calculate v and dv/dt at $t = 0^+$.

c) Calculate the voltage response for $t \geq 0$.

d) Plot $v(t)$ versus t for the time interval $0 \leq t \leq 11$ ms.



1

Solution

a) Because

$$\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad/s},$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{10^6}{(8)(0.125)}} = 10^3 \text{ rad/s},$$

we have

$$v(0^+) = V_0 = B_1$$

$$\omega_0^2 > \alpha^2.$$

$$\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2$$

2 Therefore, the response is underdamped. Now,

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 4 \times 10^4} = 100\sqrt{96}$$

$$= 979.80 \text{ rad/s},$$

$$s_1 = -\alpha + j\omega_d = -200 + j979.80 \text{ rad/s},$$

$$s_2 = -\alpha - j\omega_d = -200 - j979.80 \text{ rad/s}.$$

solving for s_1, s_2 to emphasis s_1, s_2 complex conj.

For the underdamped case, we do not ordinarily solve for s_1 and s_2 because we do not use them explicitly. However, this example emphasizes why s_1 and s_2 are known as complex frequencies.

b) Because v is the voltage across the terminals of a capacitor, we have

$$v(0) = v(0^+) = V_0 = 0.$$

Because $v(0^+) = 0$, the current in the resistive branch is zero at $t = 0^+$. Hence the current in the capacitor at $t = 0^+$ is the negative of the inductor current:

$$i_C(0^+) = -(-12.25) = 12.25 \text{ mA}.$$

Therefore the initial value of the derivative is

$$\frac{dv(0^+)}{dt} = \frac{(12.25)(10^{-3})}{(0.125)(10^{-6})} = 98{,}000 \text{ V/s.} = \frac{i_c(0)}{C}$$

we use α and ω_d

$$v(t) = e^{-\alpha t}\left[B_1 \cos\omega_d t + B_1 \sin\omega_d t\right]$$

3 c) From Eqs. 8.30 and 8.31, $B_1 = 0$ and

$$B_2 = \frac{98{,}000}{\omega_d} \approx 100 \text{ V}.$$

Substituting the numerical values of α, ω_d, B_1, and B_2 into the expression for $v(t)$ gives

$$v(t) = 100e^{-200t} \sin 979.80t \text{ V}, \quad t \geq 0.$$

d) Figure 8.9 shows the plot of $v(t)$ versus t for the first 11 ms after the stored energy is released. It clearly indicates the damped oscillatory nature of the underdamped response. The voltage $v(t)$ approaches its final value, alternating between values that are greater than and less than the final value. Furthermore, these swings about the final value decrease exponentially with time.