Probabilistic Method In Electrical Engineering

- Probability is the study of random or non-deterministic experiments.
- Probability applications arises in most science and engineering
- People some time applies probability without knowing it

Signals and systems of interest to engineers are <u>not deterministic</u>.

–You cannot predict behavior precisely.

–Deterministic analysis and design methods <u>usually don't work!</u>

Signals Like:

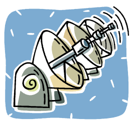
$$x(t) = a\cos(\omega t + \phi), \qquad y(t) = a\,u(t)\,e^{-\alpha t}$$

$$z[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{kn2\pi}{N}}, \qquad w[n] = \delta[n] + 4\delta[n-3]$$

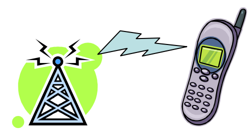
are rare

Why is a deterministic model usually inadequate?

- **EEs focus on <u>information</u> transmission and processing.**
  - –There is <u>no</u> information in deterministic signals.

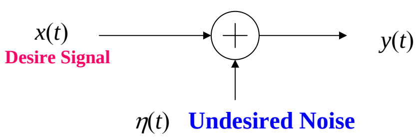


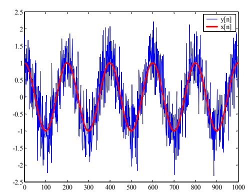
Information = Uncertainty



–Let $x(t)$ be a talk radio broadcast. How useful is it if $x(t)$ is known?

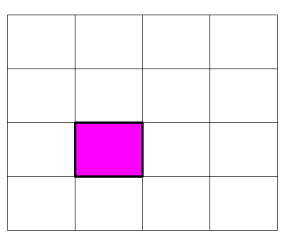
- **Noise is ubiquitous (كلية الوجود, وجود الشىء في كل مكان)**





People some time applies probability without knowing it

Example : Buying a land in a block of land



Question : What is the **approximate** price you should pay ?

Answer : You will ask about the prices of the other lands in the block.

Basically what are you doing is averaging .

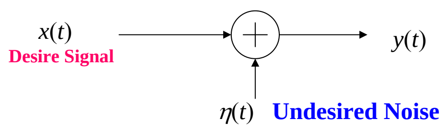
Other Applications of Probabilities are :

- Weather forecast (Temperature, Humidity, Pressure) we hear it every day in the news (درجات الحرارة المتوقعه)
- Stock prices (أسعار الأسهم) change randomly.

  If you know how the prices changes **exactly**

  pleas let me know.

- In communication system, we have noise which is undesired signal which change also randomly.



There are many other applications of probability in science like in physics, chemistry, astronomy, etc.

Historically probability started in predication in games like:

- Selecting a card from a deck of cards
- Tossing a dice
- Tossing a coin
- The question was "What is the probability of getting "a **Spade**" in selecting a card or "**Two dots**" in tossing a dice or "**Head**" in tossing a coin

The question is how to assign probability ?

- How to give a numeric value to the probability of some events?
- Common sense tells us that the probability of getting "**Head**" in tossing of a fair **coin experiment is** 1/2 or 1 out of two possibilities.
- Also the probability of getting "**Two dots**" in tossing a fair dice 1/6 or 1 out of six possibilities.

Let us put the tossing of a dice experiment in a diagram as shown

- Therefore the event " **Two dot**" is one event among other events:

  { "One dot" , " **Two dot**" , " Three dot", " Four dot", "Five dot", " Six dot" }
- Then we can **think** of an event as a set among other sets.
- Therefore it is important to know and understand set theory for probability calculations.

Set

Set definitions: A set is a collection of objects or elements.

Examples :
- A set of chairs
- A set of airplanes
- A set of integers from 0 to 5 { 0,1,2,3,4,5}

The set can be finite as the set of integers from 0 to 5

or can be countable infinite like the set of positive integers {0, 1, 2, 3, … }

The set can be countable or uncountable

Countable set you can count the members of the set as 1,2,3,…

Example : the set of the even positive numbers are countable

$$\begin{array}{cccccccc} \{ 0, & 2, & 4, & 6, & 8, & 10, & \cdots & \} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \{ 1, & 2, & 3, & 4, & 5, & 6, & \cdots & \} \end{array}$$

Uncountable set when you can not count the members of the set

Example : The set of numbers between 0 and 1 $\{ 0 < x < 1\}$

The empty set is given the symbol $\varnothing$

For a set A of N elements , how many possible subsets of A ?

Answer : There exist 2^N possible subset of A

Example : Let $A = \{ 1, 2, 3 \}$ then the possible subset of A are

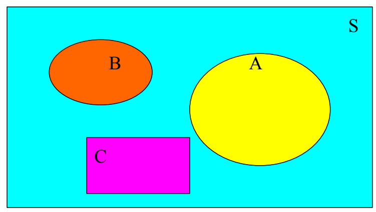
$\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, $\varnothing$

$\Rightarrow$ there are 8 subsets or 2^3

Venn Diagram

Is a representation of sets as closed-plane figures

The largest set the encompassing set of objects is called the universal set and denoted S



Consider the experiment of rolling the dice and observing the numbers shown on the upper face

Here the universal set $S = \{1, 2, 3, 4, 5, 6\}$

If every element of a set A is also an element in another set B

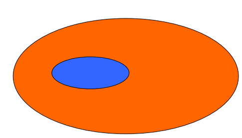
A is said to be contained in B or A is a subset of B

$$A \subseteq B$$

If at least one element exists in B which is not in A ,

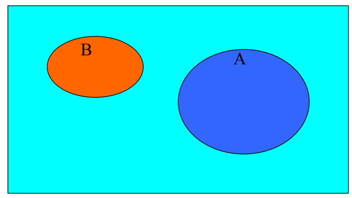
then A is a proper subset of B , denoted as

$$A \subset B$$



The null set is subset of all other sets $\varnothing$

Two sets A and B are **mutually exclusive** if they have no common elements



Example Let the sets A and B be given as

A = { 1, 3, 5, 7 } and B = { 2, 4, 6, 8, 10, 12, 14 }

SET OPERATIONS

Equality : Two set A and B are equal if all elements in A are present in B and all elements in B are present in A

$$\Rightarrow \quad A \subseteq B \text{ and } B \subseteq A$$

For equal sets we write A = B

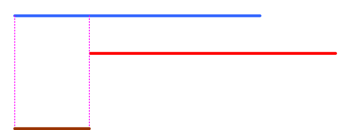
Difference : Consider two sets A and B

Denote the set $A - B$, is the set containing all elements of A that are not presents in B

Example :

Let $A = \{ 0.6 < a \le 1.6 \}$ and $B = \{ 1.0 \le b \le 2.5 \}$

Then $A - B = \{ 0.6 < c < 1.0 \}$

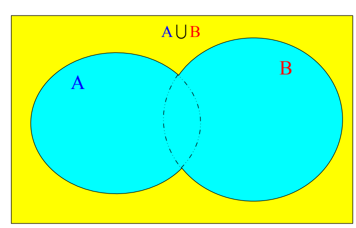


Similalrly $B - A = \{ 1.6 < d < 2.5 \} \ne A - B$

Union : The union of two sets A and B call it C is written

$C = A \cup B$

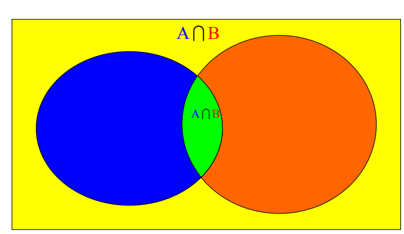
Sometimes called the sum of two sets



Intersection : The intersection of two sets A and B call it D is written

$D = A \cap B$

Sometimes called the products of two sets



The union and intersection can be repeated for N sets

$$C = A_1 \cup A_2 \cup \ \ldots \ \cup A_N = \bigcup_{i=1}^{N} A_i$$

$$D = A_1 \cap A_2 \cap \ \ldots \ \cap A_N = \bigcap_{i=1}^{N} A_i$$

Problem: Define probability in a mathematically rigorous and useful way

- A six sided die is rolled twice. What is the "probability" of the sum of the two rolls being: 1, 2, 5, 7, or 11?
- We must define what we mean by "probability, but it must be consistent with our experience.
- **1. Classical theory**, (ratio of total to favorable outcomes)
  - $P(E) = \frac{N_E}{N}$
  - Just carefully count them up!:

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  - Count them up:

2nd die	1st die 1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

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- **1. Classical theory,** (ratio of total to favorable outcomes)
  – $$P[E] = \frac{N_E}{N}$$
  – Count them up

1/36

6/36

4/36

2/36

2nd	1st die					
die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
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Problem: Define probability in a mathematically rigorous and useful way

- Advantages:
  – Clear, mathematically sound definition.
  – Based on model of the events rather than empirical observations of experimental outcomes.
- Disadvantages:
  – Assumes equally likely outcomes
    - What about loaded dice? No way to handle it!
  – Requires detailed enumeration of all outcomes. What do you do with continuous data?
- **2. Relative Frequency of Occurrence**:
  – $$P[E] = \lim_{n \to \infty} \frac{n_E}{n}$$
  – Roll the die n = 100-10,000 times, count them up! (classroom experiment, histogram).

Problem: Define probability in a mathematically rigorous and useful way

- Advantages:
  - Automatically accounts for non-uniform outcome probabilities.
  - Can handle continuous outcomes.
- Disadvantages:
  - We don't have all day! ($n << \infty$)
  - Thus, $P[E]$ is not known exactly!
    - But the probability that $\left|\frac{n_E}{n} - P[E]\right| < \delta$ can be made arbitrarily small as $n \to \infty$.
- 3. Probability Based on Axiomatic Theory
  - The modern, elegant approach to probability.

Sets, experiments and events

- Set: A collection of objects. e.g. A, B, E, F, Ω.
- Subset: A set contained within another set. e.g. $A \subset B$
- Null, or empty set: ϕ.
- Experiment: A random activity, denoted as $\mathscr{H}$ with outcomes contained in the set Ω.
- Universal set or certain event: Ω. Set of all possible outcomes.
- Outcome: A single element ζ in Ω., i.e. $\zeta \in \Omega$.
- Event: A subset, E, of Ω, i.e. the occurrence of some selected range of outcomes.

Probability

We define probability in two ways:

First is based on set theory and the fundamental axioms of probability

Second is based on relative frequencies

Probability based on set theory and axiom of probability

Basic to the study of probability is the idea of a physical experiment. A single performance of the experiment is called a "trial" for which there is an "outcome"

Example: consider the experiment of rolling a single die and observing the number of dots that show up

$\Rightarrow$ There are six numbers namely $\{1,2,3,4,5,6\}$

were 1 represent the outcome when the number of dots is 1 and 2 represent the number of dots are two and so on.

The set $\{1,2,3,4,5,6\}$ represent the set of all possible outcome of the experiment.

This set contain every possible set in the experiment

This set is called the " **sample space**" and given the symbol S.

This sample space S is the universal set we define previously

The sample space can be discrete or continuous finite or infinite countable or uncountable

Examples :

Selecting a number from the set {1,2,3,4} , this set is discrete and finite

Selecting a positive integer $\Longrightarrow$ $S = \{1, 2, 3, \cdots\}$

This set is discrete , infinite and countable

Selecting a number from the set (0,1) $\Longrightarrow$ $S = \{x \mid x \, \varepsilon \, (0,1)\}$

Events

Definition

An event is a subset of the sample space

We say that event A occur if the observed outcome of the experiment is an element of A

Example

Consider the experiment of rolling a die and observing the number of dots, here

$S = \{1, 2, 3, 4, 5, 6\}$

Let the event A be getting an even number, then any of the following will satisfy the event A namely {2}, {4}, {6}

$\Longrightarrow$ A = " getting 2" or " getting 4" or " getting 6"

$\Longrightarrow$ $A = \{2\} \cup \{4\} \cup \{6\} = \{2, 4, 6\}$

Later we will assign a probability to the event A as $P(A) = \frac{3}{6} = \frac{1}{2}$

This experiment has discrete finite outcome

All events are discrete and finite

Consider the experiment of selecting a positive integer

$$\Rightarrow \quad S = \{1, 2, 3, \cdots\}$$

The sample space is infinite countable

The event A of selecting an odd integer $\Rightarrow$ $A = \{1, 3, 5, \cdots\}$

$\Rightarrow$ is discrete infinite countable event (set)

The experiment can produces continues uncountable infinite set

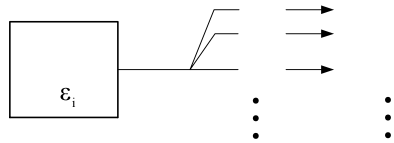
Example: consider the experiment of selecting a number between 6 and 13

$$\Rightarrow \quad S = \{ 6 \le s \le 13 \}$$

Define the event A of selecting a number greater than 7.4 and less than 7.6

$$\Rightarrow \quad A = \{ 7.4 < a < 7.6 \}$$

To each event defined on the sample space S, we shall assign a number called probability



Therefore probability maps or a function of the events to the real line (Number)
To assign these probabilities or numbers, it must satisfies three **axioms** called the **axioms of probability**

Random
Experiment
With outcome

Probability Definition and Axioms

axiom 1 : $P(A) \geq 0$ represent our desire to work with nonnegative numbers

axiom 2 : $P(S) = 1$ The sample space is an event with the highest probability , citrine event

$P(\varnothing) = 0$ The null event is an event with the lowest probability , impossible event

axiom 3 $P\left(\bigcup_{n=1}^{N} A_N\right) = \sum_{n=1}^{N} P(A_N)$ if $A_m \cap A_n = \varnothing$

The probability of an event which is the union of mutually **exclusive events** is equal to the sum of the individual event probabilities

Joint Probability

In some experiments , some events are not mutually exclusive, these events have common elements in the sample space

Example : in the die experiment,
the sample space $S = \{1, 2, 3, 4, 5, 6\}$

Let the events A and B define as
$A = \{1, 2, 3, 4\}$
$B = \{3,4,5\}$
Hence A and B are not mutually exclusive

$$\Rightarrow P(A \cap B) = \{1, 2, 3, 4\}$$

now suppose we want to find the probability of $P(A \cup B)$

$$P(A \cup B) = P(\{1, 2, 3, 4\} \cup \{3, 4, 5\})$$

$$= P(\{1, 2, 3, 4\}) + P(\{3, 4, 5\}) - P(\{3, 4\})$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In general $P(A \cup B) \leq P(A) + P(B)$

For mutually exclusive events, we have

$$A \cap B = \varnothing \quad \Rightarrow P(A \cap B) = 0$$

Conditional Probability

Let A and B be two events , then we define the conditional probability of an event A, given B by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0$$

Conditional probability is a defined quantity and cannot be proven. However we will do some interpretation as follows:

the sample space S has n elements

event B has n_B elements

event A has n_A elements

Now if event B is given or happened, this will reduce the unknown random chance from

n to n_B

Then the probability of A given B ,

$$P(A|B) = \frac{n_{AB}}{n_B} = \frac{n_{AB}/n}{n_B/n} = \frac{P(A \cap B)}{P(B)}$$

Example

The probability $P(A|B)$ simply reflects the facts that Probability of an events A may depend on a second event B.

If A and B are mutually exclusive then

$$P(A \cap B) = P(\varnothing) = 0$$

$$\Rightarrow \ P(A|B) = 0$$

what that mean is when one event happen in mutually exclusive event, the other event can not happen (impossible to happen).

In the rolling of a die experiment, the elementary events $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$ are mutually exclusive

then $P(\{1\}|\{2\}) = 0$

since when $\{2\}$ occur or happen , it is impossible for $\{1\}$ to occur.

The conditional Probability satisfies the axiom of Probability :

(1) $P(A|B) \geq 0$

(2) $P(S|B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(3) Assume A and C to be mutually exclusive events

$$\Rightarrow \quad A \cap C = \varnothing$$

$$P\left[(A \cup C) \cap B\right] = P\left[(A \cap B) \cup (C \cap B)\right]$$

since $(A \cap B) \cap (C \cap B) = \underbrace{A \cap C}_{\varnothing} \cap B = \varnothing \cap B = \varnothing$

$\Rightarrow$ $(A \cap B)$ and $(C \cap B)$ are mutually exclusive events

$$\Rightarrow \quad P\left[(A \cup C) \cap B\right] = P(A \cap B) + P(C \cap B)$$

$$\Rightarrow P\left[(A \cup C)|B\right] = \frac{P\left[(A \cup C) \cap B\right]}{P(B)} = \frac{P(A \cap B)}{P(B)} + \frac{P(C \cap B)}{P(B)}$$

axiom 3 holds

Example 1.4-1

Example 1.4-1

A Box contain 100 resistors having resistance and tolerance as shown

	5%	10%	**Total**
22 Ω	10	14	24
47 Ω	28	16	44
100 Ω	24	8	32
Total	62	38	100

Define three events

A draw a 47 Ω

B draw a resistor with 5% tolerance

C draw a 100 Ω

$$P(A) = P(47\Omega) = \frac{44}{100} \qquad P(B) = P(5\%) = \frac{62}{100} \qquad P(C) = P(100\Omega) = \frac{32}{100}$$

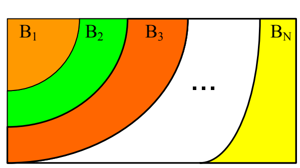
$$P(A \cap B) = P(47\Omega \ \cap \ 5\%) = \frac{28}{100} \qquad P(A \cap C) = P(47\Omega \ \cap \ 100\Omega) = 0$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{28/100}{62/100} = \frac{28}{62}$$

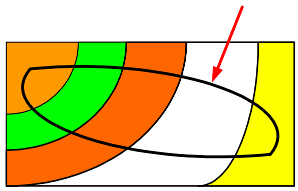
Total Probability

Let $B_n \quad n = 1,2,\ldots,N$ be N mutually exclusive events whose union equals S ,

$$\mathrm{B_m} \cap \mathrm{B_n} = \varnothing \qquad \mathrm{m} \neq \mathrm{n} = 1,2,\ldots,\mathrm{N} \qquad \bigcup_{n=1}^{N} \mathrm{B_n} = \mathrm{S}$$



Let **A** be an event defined on the sample space

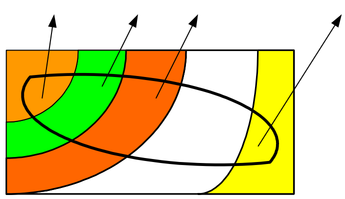


Let us find the probability of A , P(A) as follows :

A

B_1 B_2 B_3 B_N

...



$$A = A \cap S = A \cap \left(\bigcup_{n=1}^{N} B_n \right) = \bigcup_{n=1}^{N} (A \cap B_n)$$

Now the events $A \cap B_n$ are mutually exclusive events

$$(A \cap B_i) \cap (A \cap B_j) = A \cap \underbrace{B_i \cap B_j}_{\varnothing} = A \cap \varnothing = \varnothing$$

By applying axiom 3

$$P(A) = P(A \cap S) = P\left[\bigcup_{n=1}^{N} (A \cap B_n) \right] = \sum_{n=1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(A \mid B_n) P(B_n)$$

which is the total probability of event A

Bayes Theorem (After English Mathematician)

Let A and B be two events, then given B ,
the probability of A is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0$$

Now suppose we want to calculate the conditional probability P(B|A)

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \qquad P(A) > 0$$

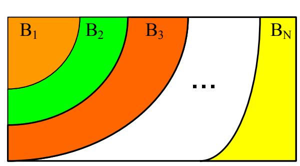
But $\quad P(B \cap A) = P(A \cap B) = P(A|B)P(B)$

$$\Rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)} \qquad P(A) > 0$$

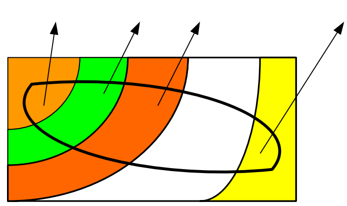
This is called the Bayes Theorem

Let $B_n \quad n = 1, 2, \ldots, N$ be N mutually exclusive events whose union equals S ,

$$B_m \cap B_n = \varnothing \qquad m \neq n = 1,2,\ldots,N \qquad \bigcup_{n=1}^{N} B_n = S$$



Let A be an event defined on the sample space



Let B_n be one of the events defined above , then we can write the conditional probability

$$P(B_n|A) = \frac{P(B_n \cap A)}{P(A)} \qquad P(A) > 0$$

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{\underbrace{P(A|B_1)P(B_1) + \ldots + P(A|B_N)P(B_N)}_{\text{Total Probabilty}}}$$

Example 1.4-2

Independent Events

Let A and B be two events with none zero probabilities of occurrence

$$\Rightarrow \quad P(A) \neq 0 \ \text{ and } \ P(B) \neq 0$$

We call the events statistically independent if the probability of occurrence of one event is not affected b y the occurrence of the other event

$\Rightarrow$ mathmatically $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Since

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \quad P(A \cap B) = P(B)P(A)$$

Therefore two events A and B are statistically independent iff

$$P(A \cap B) = P(B)P(A)$$

If two events A and B have their joint probabilities equal the product of their probability then they are statistically independent

Since two mutual events have null intersection

$$A \cap B = \varnothing \qquad \Rightarrow \; P(A \cap B) = 0$$

And since $P(A) \neq 0$ and $P(B) \neq 0$

Then two mutual events can not be independent

For two events two be independent they must have an intersection

$$\Rightarrow A \cap B \neq \varnothing$$

If there are more than two events, then the events are statistically independent **iff** they are independent by pairs

Example:

Let A_1 , A_2 and A_3 be three events , then the events are statistically independent **iff** they satisfy the four equations:

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2)P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

More generally , for N events A_1 , A_2 , … , A_N to be statistically independent iff they satisfy the following equations

$$P(A_i \cap A_j) = P(A_i)P(A_j)$$
$$P(A_i \cap A_j \cap A_k) = P(A_i)P(A_j)P(A_k)$$
$$\vdots$$
$$P(A_1 \cap A_2 \cap \ldots \cap A_N) = P(A_1)P(A_2) \ldots P(A_N)$$

be satisfied for all $1 \le i < j < k < \cdots \le N$

Combined Experiment

Most of the experiment we considered so far were the outcomes from a single experiment (except the example we considered about tossing the die twice).

Most practical problems arises by combining several experiment to get a combined experiment, examples :

- The simultaneous measurement of "pressure", "wind speed" and "temperature" at some location at an instant of time.
- Flipping a coin N times, **example twice** $S = \{HH,HT,TH,TT\}$
- Tossing a die several time, example twice
  $S = \{(1,1), (1,2) \ldots (1,6), (2,1) \ldots (6,6)\}$

Therefore a combined experiment consist of forming a single experiment by suitably combing individual experiments which we call sub experiment.

We recall that an experiment is defined by specifying:

1. The sample space S
2. The events defined on the sample space
3. The probabilities of the events defined on the sample space

We specify the three quantities that define an experiment (sample space, events and probabilities of the events) for a combined experiment.

We are going to use the rolling of a die twice experiment

(1) Combined Sample Space

Let S_1 and S_2 be the sample spaces of the two sub experiment of rolling the die twice :

$S_1 = \{1, 2, 3, 4, 5, 6\}$ $S_2 = \{1, 2, 3, 4, 5, 6\}$

Now the combine sample space of the experiment of rolling the die twice :

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The combined sample space is denoted as

$S = S_1 \text{ x } S_2$

the element (1,6) is different from (6,1)
Then S is an order pairs of objects

Other combined experiment sample space :

Flipping a coin $\Rightarrow$ $S_1 = \{H, T\}$

Rolling a single die $\Rightarrow$ $S_2 = \{1, 2, 3, 4, 5, 6\}$

The combine sample space

$$\Rightarrow S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$$

In general if there are N sub experiments with sample spaces S_n , n = 1, 2, … N , the combined sample space is

$$S = S_1 \times S_2 \times S_3 \cdots \times S_N$$

(2) Events on the Combined Space

Events defined on the combined sample space through their relationship with events defined on the sub experiment sample space.

Example: on the rolling of the die twice experiment

Let A and B be events defined on S_1 and S_2 given as

$A = \{1, 2, 3\}$ and $B = \{1, 2\}$

then define an event C as $C = A \times B$

$\Rightarrow C = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$ is an event defined on S

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

An event also can be the union and intersection of other evens.

Consider the event A of getting sum 7 in the experiment of rolling a die twice

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$= \{\underbrace{(1,6)}_{\{1\}x\{6\}} \cup \underbrace{(2,5)}_{\{2\}x\{5\}} \cup \underbrace{(3,4)}_{\{3\}x\{4\}} \cup \underbrace{(4,3)}_{\{4\}x\{3\}} \cup \underbrace{(5,2)}_{\{5\}x\{2\}} \cup \underbrace{(6,1)}_{\{6\}x\{1\}}\}$$

In general if there are N sub experiments with sample spaces S_n , n = 1, 2, … N , on which events A_n are defined, the events on the combined sample space S will be all be sets of the form

$$A_1 \text{ x } A_2 \text{ x } A_3 \cdots \text{ x } A_N$$

and union and intersections of such sets

(3) The probabilities of the events defined on the sample space

Considering the sub experiment to be independent sub experiments as on the rolling of die twice.

If A and B are events defined on S_1 and S_2

$$A \subset S_1 \quad \text{and} \quad B \subset S_2$$

then $P(A \times B) = P(A)P(B)$

Example :

Consider the event A of getting sum 7 in the experiment of rolling a die twice

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$= \{\underbrace{(1,6)}_{\{1\}x\{6\}} \cup \underbrace{(2,5)}_{\{2\}x\{5\}} \cup \underbrace{(3,4)}_{\{3\}x\{4\}} \cup \underbrace{(4,3)}_{\{4\}x\{3\}} \cup \underbrace{(5,2)}_{\{5\}x\{2\}} \cup \underbrace{(6,1)}_{\{6\}x\{1\}}\}$$

$$\Rightarrow P(A) = P\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$= P\left\{\begin{matrix} \{\{1\}x\{6\}\} \cup \{\{2\}x\{5\}\} \cup \{\{3\}x\{4\}\} \\ \cup \{\{4\}x\{3\}\} \cup \{\{5\}x\{2\}\} \cup \{\{6\}x\{1\}\} \end{matrix}\right\}$$

$$= P\{1\}P\{6\} + P\{2\}P\{5\} + P\{3\}P\{4\}$$
$$+ P\{4\}P\{3\} + P\{5\}P\{2\} + P\{6\}P\{1\}$$

$$= \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6}$$

$$= \frac{1}{6}$$

For N independent sub experiments S_n n= 1,2, … N , the generalization

$$P(A_1 \times A_2 \times \dots \times A_N) = P(A_1)P(A_2) \dots P(A_N)$$
where $A_n \subset S_n$, n = 1, 2, ..., N.

Some experiment involve repeated trial in which the outcomes are elements of a finite sample space and they are not replaced after each trial

Example :

Drawing four cards from a deck of 52-cards deck , each of the "draws" is not replaced, so draws has the followings samples spaces (i.e, number of possibilities)

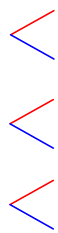
52 for the first draw
51 for the second draw
50 for the third draw
49 for the fourth draw

The number of different cards that you select (4 cards out of 52) are

(52)(51)(50)(49) = **6,497,400**

Let us consider the following example

A box contain three numbered balls {1,2,3} , two balls are selected without replacements , we get the following events or sets



here the number of elements generated from taken 2 elements from 3 is 6 or (3)(2)

In general the number of r elements taken from a sample of n elements without replacements when order is important is

$$\left.\begin{array}{l}\text{Orderings of } r \text{ elements}\\ \text{taken from } n\end{array}\right\} = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n!}{(n-r)!} = P^{n}_{r}, \quad r = 1, 2, \ldots, n$$

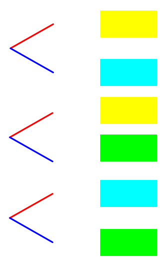
This number is the number of *permutations* or sequences of r elements taken from n elements when <u>order is of occurrence is important.</u>

1 2
3
2 1
3
3 1
2

If the order is not important , example in the selecting 2 numbers out of 3

(1,2) is the same as (2,1)

then the number of possible sequence is less



In the example above then the number of possible elements you get from selecting 2 numbers out of 3 numbers is 3

The number of *permutations* is reduced by 2 or 2!

This number of sequences when the order is not important is called the number of *combinations* of r things taken from n things and is given as

$$\left.\begin{array}{l}\text{Number of } r \text{ elements} \\ \text{taken from } n \text{ when order} \\ \text{is not important}\end{array}\right\} = \frac{\text{Number of permutations } P^n_r}{r!}$$

$$= \frac{P^n_r}{P^r_r}$$

There is a special notation for the number *combinations* of r things taken from n elements which is given as

$$\binom{n}{r} = \frac{P^n_r}{P^r_r} = \frac{n!}{(n-r)!\,r!}$$

The numbers of *combinations* $\binom{n}{r}$ are called binomial coefficients because they are central to the expansion of the binomial $(x+y)^n$ as given by

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

we define $0! = 1$, $\Rightarrow \quad \binom{n}{0} = 1 \qquad \binom{n}{n} = 1$

Bernoulli Trials

Consider the experiment of tossing a fair coin 3 times then the sample space of the experiment is

$$S = S_1 \ x \ S_2 \ x \ S_3$$

where $S_1 = \{H, T\}$

$S_2 = \{H, T\}$

$S_3 = \{H, T\}$

which produce the following sampling space $S = \begin{Bmatrix} H & H & H \\ H & H & T \\ H & T & H \\ H & T & T \\ T & H & H \\ T & H & T \\ T & T & H \\ T & T & T \end{Bmatrix}$

Now let us find the probability of getting lat say "Two Heads"

The sample space S contain 8 elements out of that 8 elements there are 3 elements that contain 2 heads

$$S = \begin{Bmatrix} H & H & H \\ H & H & T \\ H & T & H \\ H & T & T \\ T & H & H \\ T & H & T \\ T & T & H \\ T & T & T \end{Bmatrix} \begin{matrix} \\ \Leftarrow \\ \Leftarrow \\ \\ \Leftarrow \\ \\ \\ \\ \end{matrix}$$

Therefore P("Two Heads") = 3/8

Now let us find that probability using the combinations method as follows:

On the above experiment, we have the number of times we get 2 heads in the tossing of the coin is 3 times

$$\binom{3}{2} = \frac{3!}{(3 - 2)!\, 2!} = \frac{(3)(2)(1)}{(1)(2)(1)} = 3$$

$$S_{2\ \text{Head}} = \begin{Bmatrix} H & H & T \\ H & T & H \\ T & H & H \end{Bmatrix}$$

Assuming that the coin is fair then

$$P(H) = P(T) = \frac{1}{2}$$

Since the probability of any of the sequences in $S_{2\ \text{Head}}$ are equal:

$$P(HHT) = P(HTH) = P(THH) = (\underbrace{\tfrac{1}{2}}_{\text{2-Heads}})^2 \ (\underbrace{1-\tfrac{1}{2}}_{\text{1-Tail}})^{(3-2)}$$

$$= (\tfrac{1}{4})(\tfrac{1}{2}) = \tfrac{1}{8}$$

Since there are 3 sequences of two heads and one tail then

$$P(S_{2\ \text{Head}}) = \frac{3}{8}$$

In general in an experiment were the outcome say A is one out of two , then the other out come will be $\bar{A}$ such as

Flipping a coin , Head or Tail

Hitting or Missing a target

Sending a 0 or 1

Assume the probability of A is p

$$\Rightarrow P(A) = p \qquad \text{and} \qquad P(\bar{A}) = 1 - p = q$$

Now suppose we conduct this experiment (Flipping a coin , Firing on a target or Sending 0 and 1) N times and we are interested in the probability of getting the event A k-times out of this N-trial.

There will be $\binom{N}{k}$ possible sequence and each sequence will have the event A occurring k-time and the event $\bar{A}$ occurring $N-k$

Then the probability of the event A occurring k-time and event $\bar{A}$ occurring N-k times is

$$\underbrace{P(A)P(A)\cdots P(A)}_{k \quad \text{terms}}\underbrace{P(\bar{A})P(\bar{A})\cdots P(\bar{A})}_{N-k \quad \text{terms}} = p^{k}(1-p)^{N-k}$$

since there are $\binom{N}{k}$ possible sequence then

$$P\begin{Bmatrix} \text{A occurs exactly } k \text{ times} \\ \text{out of } N \text{ trial} \end{Bmatrix} = \binom{N}{k} p^{k}(1-p)^{N-k}$$

Example 1.7-1

Submarine (غواصه) attempts to sink aircraft carrier (حاملة طائرات)

The submarine will successfully sink the aircraft carrier if **two or more torpedoes hit** the carrier. The submarine fires three torpedoes

The probability of a **hit** (**success**) is **0.4** for each torpedo

What is the probability that the carrier will be sunk ?

Solution: Assume M torpedo miss and H torpedo hit

MMM → No hit $\Rightarrow \binom{3}{0} = \dfrac{3!}{0!(3-0)!} = 1$

MMH, MHM, HMM → One hit $\Rightarrow \binom{3}{1} = \dfrac{3!}{1!(3-1)!} = 3$

MHH, HMH, HHM → Two hits $\Rightarrow \binom{3}{2} = \dfrac{3!}{2!(3-2)!} = 3$

HHH → Three hit $\Rightarrow \binom{3}{3} = \dfrac{3!}{3!(3-3)!} = 1$

P{ carrier sunk} = P{ two or more hits }

= P{ exactly two hits} + P{ exactly three hits}

$$P\{\text{ exactly two hits}\} = \binom{3}{2}(0.4)^2(1-0.4)^1 = 0.288$$

$$P\{\text{ exactly three hits}\} = \binom{3}{3}(0.4)^3(1-0.4)^0 = 0.064$$

P{ carrier sunk} = 0.288 + 0.064 = 0.352